# Chapter 8
# Binomial Theorem

## Exercise 8.1

**Question 1**

**Expand each of the expressions in Exercises 1 to 5.**
**$(1 – 2x)^5$**

**Answer:**

From binomial theorem expansion we can write as
$(1 – 2x)^5$
$= {}^5C_0\,(1)^5 - {}^5C_1\,(1)^4\,(2x) + {}^5C_2\,(1)^3\,(2x)^1 - {}^5C_3\,(1)^3\,(2x)^3 + {}^5C_4\,(1)^1\,(2x)^4 - {}^5C_5\,(2x)^5$
$= 1 – 5\,(2x) + 10\,(4x)^2 – 10\,(8x^3) + 5\,(16\,x^4) – (32\,x^5)$
$= 1 – 10x + 40x^2 – 80x^3 – 32x^5$

**Question 2**

$\left(\frac{2}{x} - \frac{x}{2}\right)^5$

**Answer:**

From binomial theorem given equation can be expanded as
$\left(\frac{2}{x} - \frac{x}{2}\right)^5 = {}^5C_0\left(\frac{2}{x}\right)^3 - {}^5C_1\left(\frac{2}{x}\right)^4\left(\frac{x}{2}\right) + {}^5C_2\left(\frac{2}{x}\right)^3\left(\frac{x}{2}\right)^2$
$- {}^3C_3\left(\frac{2}{x}\right)^2\left(\frac{x}{2}\right)^3 + {}^3C_4\left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^4 - {}^3C_5\left(\frac{x}{2}\right)^5$
$= \frac{32}{x^5} - 5\left(\frac{16}{x^4}\right)\left(\frac{x}{2}\right) + 10\left(\frac{8}{x^3}\right)\left(\frac{x^2}{4}\right) - 10\left(\frac{4}{x^2}\right) + 5\left(\frac{2}{x}\right)\left(\frac{x^4}{16}\right) - \frac{x^5}{32}$
$= \frac{32}{x^5} - \frac{40}{x^3} + \frac{20}{x} - 5x + \frac{5}{8}x^3 - \frac{x^3}{32}$

**Question 3**

$(2x - 3)^6$

**Answer:**

$(2x - 3)^6 = {}^6C_0\,(2x)^6 - {}^6C_1\,(2x)^5\,(3) + {}^6C_1\,(2x)^4\,(3)^2 - {}^4C_3\,(2x)^3\,(3)^3$

$= 64x^6 - 6\,(32x^5)\,(3) + 15\,(16x^4)(9) - 20\,(8x^3)\,(27)$
$+ 15\,(4x)^2\,(81) - 6\,(2x)(243) + 729$
$= 64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$

**Question 4**

$\left(\frac{x}{3} + \frac{1}{x}\right)^5$

**Answer:**

$\left(\frac{x}{3} + \frac{1}{x}\right)^5 = {}^5C_o \left(\frac{x}{3}\right)^5 + {}^3C_1 \left(\frac{x}{3}\right)^4 \left(\frac{1}{x}\right) + {}^3C_2 \left(\frac{x}{3}\right)^3 \left(\frac{1}{x}\right)^2$
$= \frac{x^5}{243} + 5\left(\frac{x^4}{80}\right)\left(\frac{1}{x}\right) + 10\left(\frac{x^3}{27}\right)\left(\frac{1}{x^2}\right) + 10\left(\frac{x^2}{9}\right)\left(\frac{1}{x^3}\right) + 5\left(\frac{x}{3}\right)\left(\frac{1}{x^4}\right) + \frac{1}{x^5}$
$= \frac{x^5}{243} + \frac{x^5}{243} + \frac{10x}{27} + \frac{10}{9x} + \frac{5}{3x^3} + \frac{1}{x^3}$

**Question 5**

$\left(x + \frac{1}{x}\right)^6$

**Answer:**

From binominal theorem, given can be expanded as
$\left(x + \frac{1}{x}\right)^6 = {}^6C_o\,(x)^6 + {}^6C_1\,(x)^1 + \left(\frac{1}{x}\right) + {}^6C_2\,(x)^4 \left(\frac{1}{x}\right)^2$
$+ {}^6C_3\,(x)^3 \left(\frac{1}{x}\right)^3 + {}^6C_4\,(x)^2 \left(\frac{1}{x}\right)^4 + {}^6C_3\,(x) \left(\frac{1}{x}\right)^5 + {}^6C_6 \left(\frac{1}{x}\right)^6$
$= x^4 + 6(x)^3 \left(\frac{1}{x}\right) + 15\,(x)^4 \left(\frac{1}{x^2}\right) + 20\,(x)^3 \left(\frac{1}{x^3}\right) + 15\,(x)^2 \left(\frac{1}{x^4}\right) + 6\,(x) \left(\frac{1}{x^5}\right) + \frac{1}{x^6}$
$= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$

**Question 6**

$(96)^3$

**Answer:**

Given $(96)^3$
96 can be expressed as the sum or difference of two numbers and then binomial theorem can be applied.
The given question can be written as 96 = 100 – 4
$(96)^3 = (100 - 4)^3$
$= {}^3C_0\,(100)^3 - {}^3C_1\,(100)^2\,(4) - {}^3C_2\,(100)\,(4)^2 - {}^3C_3\,(4)^3$

$= (100)^3 – 3 (100)^2 (4) + 3 (100) (4)^2 – (4)^3$
$= 1000000 – 120000 + 4800 – 64$
$= 884736$

**Question 7**

**$(102)^5$**

**Answer:**

Given $(102)^5$
102 can be expressed as the sum or difference of two numbers and then binomial theorem can be applied.
The given question can be written as $102 = 100 + 2$
$(102)^5 = (100 + 2)^5$
$= {}^5C_0 (100)^5 + {}^5C_1 (100)^4 (2) + {}^5C_2 (100)^3 (2)^2 + {}^5C_3 (100)^2 (2)^3 + {}^5C_4 (100) (2)^4 + {}^5C_5 (2)^5$
$= (100)^5 + 5 (100)^4 (2) + 10 (100)^3 (2)^2 + 5 (100) (2)^3 + 5 (100) (2)^4 + (2)^5$
$= 10000000000 + 1000000000 + 40000000 + 80000 + 8000 + 32$
$= 11040808032$

**Question 8**

**$(101)^4$**

**Answer:**

Given $(101)^4$
101 can be expressed as the sum or difference of two numbers and then binomial theorem can be applied.
The given question can be written as $101 = 100 + 1$
$(101)^4 = (100 + 1)^4$
$= {}^4C_0 (100)^4 + {}^4C_1 (100)^3 (1) + {}^4C_2 (100)^2 (1)^2 + {}^4C_3 (100) (1)^2 + {}^4C_4 (1)^4$
$= (100)^4 + 4 (100)^3 + 6 (100)^2 + 4 (100) + (1)^4$
$= 100000000 + 400000 + 60000 + 400 + 1$
$= 1040604001$

**Question 9**

**$(99)^5$**

**Answer:**

Given $(99)^5$
99 can be written as the sum or difference of two numbers then binomial theorem can be applied.

The given question can be written as 99 = 100 -1
$(99)^5 = (100 - 1)^5$
$= {}^5C_0 (100)^5 - {}^5C_1 (100)^4 (1) + {}^5C_2 (100)^3 (1)^2 - {}^5C_3 (100)^2 (1)^3 + {}^5C_4 (100) (1)4 - {}^5C_5 (1)^5$
$= (100)^5 - 5 (100)^4 + 10 (100)^3 - 10 (100)^2 + 5 (100) - 1$
= 1000000000 - 5000000000 + 10000000 - 100000 + 500 - 1
= 9509900499

**Question 10**

**Using Binomial Theorem, indicate which number is larger $(1.1)^{10000}$ or 1000.**

**Answer:**

By splitting the given 1.1 and then applying binomial theorem, the first few terms of $(1.1)^{10000}$ can be obtained as
$(1.1)^{10000} = (1 + 0.1)^{10000}$
$= (1 + 0.1)^{10000}$ C1 (1.1) + other positive terms
$= 1 + 10000 \times 1.1$ + other positive terms
= 1 + 11000 + other positive terms
> 1000
$(1.1)^{10000} > 1000$

**Question 11**

**Find $(a + b)^4 - (a - b)^4$. Hence, evaluate $\left(\sqrt{3} + \sqrt{2}\right)^4 - \left(\sqrt{3} - \sqrt{2}\right)^4$**

**Answer:**

Using binomial theorem the expression $(a + b)^4$ and $(a - b)^4$, can be expanded
$(a + b)^4 = {}^4C_0 a^4 + {}^4C_1 a^3 b + {}^4C_2 a^2 b^2 + {}^4C_3 a b^3 + {}^4C_4 b^4$
$(a - b)^4 = {}^4C_0 a^4 - {}^4C_1 a^3 b + {}^4C_2 a^2 b^2 - {}^4C_3 a b^3 + {}^4C_4 b^4$
Now $(a + b)^4 - (a - b)^4 = {}^4C_0 a^4 + {}^4C_1 a^3 b + {}^4C_2 a^2 b^2 + {}^4C_3 a b^3 + {}^4C_4 b^4 - [{}^4C_0 a4 - {}^4C_1 a^3 b + {}^4C_2 a2 b^2 - {}^4C_3 a b^3 + {}^4C_4 b^4]$
$= 2 ({}^4C_1 a^3 b + {}^4C_3 a b^3)$
$= 2 (4a^3 b + 4ab^3)$
$= 8ab (a^2 + b^2)$
Now by substituting $a = \sqrt{3}$ and $b = \sqrt{2}$ we get
$(\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4 = 8 (\sqrt{3}) (\sqrt{2}) \{(\sqrt{3})^2 + (\sqrt{2})^2\}$
$= 8 (\sqrt{6}) (3 + 2)$
$= 40 \sqrt{6}$

**Question 12**

**Find (x + 1)6 + (x – 1)6. Hence or otherwise evaluate**
$(\sqrt{2}+1)^6 + (\sqrt{2}-1)^6$

**Answer:**

Using binomial theorem the expressions, $(x + 1)^6$ and $(x – 1)^6$ can be expressed as
$(x + 1)^6 = {}^6C_0\, x^6 + {}^6C_1\, x^5 + {}^6C_2\, x^4 + {}^6C_3\, x^3 + {}^6C_4\, x^2 + {}^6C_5\, x + {}^6C_6$
$(x – 1)^6 = {}^6C_0\, x^6 - {}^6C_1\, x^5 + {}^6C_2\, x^4 - {}^6C_3\, x^3 + {}^6C_4\, x^2 - {}^6C_5\, x + {}^6C_6$
Now, $(x + 1)^6 - (x – 1)^6 = {}^6C_0\, x^6 + {}^6C_1\, x^5 + {}^6C_2\, x^4 + {}^6C_3\, x^3 + {}^6C_4\, x^2 + {}^6C_5\, x + {}^6C_6 – [{}^6C_0\, x6 - {}^6C_1\, x^5 + {}^6C_2\, x^4 - {}^6C_3\, x^3 + {}^6C_4\, x^2 - {}^6C_5\, x + {}^6C_6]$
$= 2\, [{}^6C_0\, x^6 + {}^6C_2\, x^4 + {}^6C_4\, x^2 + {}^6C_6]$
= 2 [x6 + 15x4 + 15x2 + 1]
Now by substituting x = √2 we get
(√2 + 1)6 - (√2 - 1)6 = 2 [(√2)6 + 15(√2)4 + 15(√2)2 + 1]
= 2 (8 + 15 × 4 + 15 × 2 + 1)
= 2 (8 + 60 + 30 + 1)
= 2 (99)
= 198

**Question 13**

**Show that 9n+1 – 8n – 9 is divisible by 64, whenever n is a positive integer.**

**Answer:**

In order to show that $9^{n+1} – 8n – 9$ is divisible by 64, it has to be show that $9^{n+1} – 8n – 9 =64$ k, where k is some natural number
Using binomial theorem,
$(1 + a)^m = {}^mC_0 + {}^mC_1\, a + {}^mC_2\, a^2 + .... + {}^mC_m\, a^m$
For a = 8 and m = n + 1 we get
$(1 + 8)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1\,(8) + {}^{n+1}C_2\,(8)^2 + .... + {}^{n+1}C_{n+1}\,(8)^{n+1}$
$9^{n+1} = 1 + (n + 1)\, 8 + 82\, [n+1C2 + n+1C_3\,(8) + .... + {}^{n+1}C_{n+1}\,(8)^{n-1}]$
$9^{n+1} = 9 + 8n + 64\, [n+1C2 + n+1C3\,(8) + .... + {}^{n+1}C_{n+1}\,(8)^{n-1}]$
$9^{n+1} – 8n – 9 = 64$ k
Where $k = [{}^{n+1}C_2 + {}^{n+1}C_3\,(8) + .... + {}^{n+1}C_{n+1}\,(8)^{n-1}]$ is a natural number
Thus, $9^{n+1} – 8n – 9$ is divisible by 64, whenever n is positive integer.
Hence the proof

**Question 14**

**Prove that**

$$\sum_{r=0}^{n} 3^{r}\ {}^{n}\mathrm{C}r\ 4^{n}$$

**Answer:**

$$\sum_{r=0}^{n} \binom{n}{r}\ a^{n-r}\ b^{r}\ =\ (a+b)^{n}$$

On right side we need $4^{n}$ so we will put the values as,
Putting b = 3 & a = 1 in the above equations, we get

$$\sum_{r=0}^{n} \binom{n}{r}\ (1)^{n-r}\ (3)^{r}\ =\ (1+3)^{n}$$

$$\sum_{r=0}^{n} \binom{n}{r}\ (1)\ (3)^{r}\ =\ (4)^{n}$$

$$\sum_{r=0}^{n} \binom{n}{r}\ (3)^{r}\ =\ (4)^{n}$$

Hence proved

# Exercise 8.2

**Question 1**

**Find the coefficient of $x^5$ in $(x + 3)^8$**



**Answer:**

The general term $T_{r+1}$ in the binomial expansion is given by $T_{r+1} = {}^{n}C_{r}\, a^{n-r}\, b^{r}$
Here $x^5$ is the $T_{r+1}$ term so a= x, b = 3 and n =8
$T_{r+1} = {}^{8}C_{r}\, x^{8-r}\, 3^{r}$............... (i)
For finding out $x^5$
We have to equate $x^5 = x^{8-r}$
⇒ r= 3
Putting value of r in (i) we get
$T_{3+1} = {}^{8}C_{3} \times^{8-3} 3^{3}$
$T_4 = \frac{8!}{3!5!} \times x^5 \times 27$
= 1512 $x^5$

Hence the coefficient of $x^5$= 1512

**Question 2**

**$a^5b^7$ in $(a - 2b)^{12}$ .**

**Answer:**

The general term $T_{r+1}$ in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$
Here a = a, b = -2b & n =12
Substituting the values, we get
$T_{r+1} = {}^{12}C_r\, a^{12-r}\, (-2b)^r$………. (i)
To find $a^5$
We equate $a^{12-r} = a^5$
r = 7
Putting r = 7 in (i)
$T_8 = {}^{12}C_7\, a^5| \,(-2b)^7$
$T_8 = \frac{12!}{7!5!} \times x^5 \times (-2)^7\, b^7$
$= -101376\, a^5\, b^7$
Hence the coefficient of a5b7= -101376

**Question 3**

**Write the general term in the expansion of**
**$(x^2 - y)^6$**

**Answer:**

The general term Tr+1 in the binomial expansion is given by
$T^{r+1} = {}^nC_r\, a^{n-r}\, b^r$…….. (i)
Here $a = x^2$, n = 6 and b = -y
Putting values in (i)
$T^{r+1} = {}^6C_r\, x^{2(6-r)}\, (-1)^r\, y^r$
$= \frac{6!}{r!(6-r)!} \times x^{12-2r} \times (-1)^r \times y^r$
$= -r^1 \frac{6!}{r!(6-r)!} \times x^{12-2r} \times y^r$
$= -1^r\, {}^6C_r\, .x^{12-2r}.\, yr$

**Question 4**

**$(x^2 - y\,x)^{12}$, $x \neq 0$.**

**Answer:**

The general term $T_{r+1}$ in the binomial expansion is given by $T^{r+1} = {}^{n}C_{r}\, a^{n-r}\, b^{r}$
Here n = 12, a= $x^2$ and b = -y x
Substituting the values we get
$T_{n+1} = {}^{12}C_r \times x^{2(12-r)} (-1)^r y^r x^r$
$= \frac{12!}{r!(12-r)!} \times x^{12-2r} \times (-1)^{r} \times y^r$
$= - r^1 \frac{6!}{r!(6-r)!} \times x^{12-2r} \times y^r$
$= -1^r\ {}^{12}cr\ .x^{24-2r}.\ y^r$

**Question 5**

**Find the 4th term in the expansion(x – 2y) $^{12.}$**

**Answer:**

The general term $T_{r+1}$ in the binomial expansion is given by $T_{r+1} = {}^{n}C_{r}\, a^{n-r}\, b^{r}$
Here a= x, n =12, r= 3 and b = -2y
By substituting the values we get
$T_4 = {}^{12}C_3\, x^9\, (-2y)^3$
$= \frac{12!}{3!9!} \times x^9 \times -8 \times y^3$
$= \frac{12 \times 11 \times 10 \times 8}{3 \times 2 \times 1} \times x^9 y^3$
$= 1760\, x^9 y^3$

**Question 6**

$\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}, x \neq 0$

**Answer:**

The general term $T_{r+1}$ in the binominal expansion is given $T_{r+1} = {}^{n}C_{r}\, a^{n-r}\, b^{r}$
Here a = 9x $b = -\frac{1}{3\sqrt{x}}$ n = 18 and r = 12
Putting values
$T_{13} = \frac{8!}{12!6!} 9x^{18-12} \left(-\frac{1}{3\sqrt{x}}\right)^{12}$
$= \frac{(18 \times 17 \times 16 \times 15 \times 14 \times 13 \times 12!)}{12! \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} \times 3^{12} \times x^6 \times \frac{1}{x^6} \times \frac{1}{3^{12}}$
= 18564

**Question 7**

**Find the middle terms in the expansions of**

$\left(3-\frac{x^3}{6}\right)^7$

**Answer:**

Here n – 7 so there would be two middle terms given by
$\left(\frac{n+1^{th}}{2}\right)$ term $=4^{th}$ and $\left(\frac{n+1}{2}+1\right)^{th}$ term $=5^{th}$
We have
$a = 3, n = 7$ and $b = -\frac{x^2}{6}$
For $T_4$ r= 3
The term will be
$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$
$T_4 = \frac{7!}{3!}\, 3^4 \left(-\frac{x^3}{6}\right)^3$
$= \frac{7\times 6\times 5\times 4}{3\times 2\times 1} \times 3^4 \times \frac{x^9}{2^3\, 3^3}$
$= -\frac{105}{8}\, x^9$
For $T_5$ term, r = 4
The term $T_{r+1}$ in the binominal expansion is given by
$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$
$T_5 = \frac{7!}{4!3!}\, 3^3 \left(-\frac{x^3}{6}\right)^4$
$= \frac{7\times 6\times 5\times 4!}{4!3!} \times \frac{3^3}{2^4\, 3^4} \times x^3 = \frac{35\,x^{12}}{48}$

## Question 8

$\left(\frac{x}{3}+9y\right)^{10}$

**Answer:**

Here n is even so the middle term will be given by $\left(\frac{n+1}{2}\right)^{th}$ term $= 6^{th}$ term the general term $T_{r+1}$ in the binominal expansion is given $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$
Now $a = \frac{x}{3}$, b = 9y, n = 10 and r = 5
Substituting the values
$T_6 = \frac{10!}{5!5!} \times \left(\frac{x}{3}\right)^5 \times (9y)^5$
$= \frac{10\times 9\times 8\times 7\times 6\times 5!}{5!\times 5\times 4\times 3\times 2\times 1} \times \frac{x^5}{3^5} \times 3^{10} \times y^5$
$= 61296\, x^5y^5$

## Question 9

**In the expansion of $(1 + a)^{m+n}$, prove that coefficients of am and an are equal.**

**Answer:**

We know that the general term $T^{r+1}$ in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$
Here n= m+n, a = 1 and b= a
Substituting the values in the general form
$T_{r+1} = {}^{m+n}C_r\, 1^{m+n}\text{-}r\, a^r$
= m+n Cr ar………… (i)
Now we have that the general term for the expression is,
$T_{r+1} = {}^{m+n}C_r\, a^r$
Now, for coefficient of am
$T_{m+1} = {}^{m+n}C_m\, a^m$
Hence, for coefficient of $a^m$, value of r = m
So, the coefficient is ${}^{m+n}C_m$
Similarly, Coefficient of an is ${}^{m+n}C_n$
${}^{m+n}C_m = \frac{(m+n)!}{m!n!}$
And also, ${}^{m+n}C_m = \frac{(m+n)!}{m!n!}$
The coefficient of $a^m$ and $a^n$ are same that is $\frac{(m+n)!}{m!n!}$

## Question 10

**The coefficients of the $(r - 1)^{th}$, $r^{th}$ and $(r + 1)^{th}$ terms in the expansion of (x + 1)n are in the ratio 1 : 3 : 5. Find n and r.**

**Answer:**

The general term $T_{r+1}$ in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$
Here the binomial is $(1+x)^n$ with a = 1, b = x and n = n
The $(r+1)^{th}$ term is given by
$T_{(r+1)} = {}^nC_r\, 1^{n-r}\, x^r$
$T_{(r+1)} = {}^nC_r\, x^r$
The coefficient of $(r+1)^{th}$ term is ${}^nC_r$
The $r^{th}$ term is given by $(r-1)^{th}$ term
T(r+1-1) = nCr-1 xr-1
$T_{r-1} = {}^nC_{r-1}\, x^{r-1}$
∴ the coefficient of $r^{th}$ term is ${}^nC_{r-1}$
For $(r-1)^{th}$ term we will take $(r-2)^{th}$ term
$T_{r-2+1} = {}^nC_{r-2}\, x^{r-2}$
$T_{r-1} = {}^nC_{r-2}\, x^{r-2}$
∴ the coefficient of $(r-1)^{th}$ term is ${}^nC_{r-2}$
Given that the coefficient of $(r-1)^{th,}$ $r^{th}$ and $r+1^{th}$ term are in ratio 1:3:5

Therefore,

$$\frac{\text{the coeffiecient of } r-1^{th} \text{ term}}{\text{cofficient of } r^{th} \text{ term}} = \frac{1}{3}$$

$$\frac{{}^{n}c_{r-2}}{{}^{n}c_{r-1}} = \frac{1}{3}$$

$$\Rightarrow \frac{\frac{n!}{(r-2)!\,(n-r+2)!}}{\frac{n!}{(r-1)!\,(n-r+1)!}} = \frac{1}{3}$$

On rearranging we get

$$\frac{n!}{(r-2)!(n-r+2)!} \times \frac{(r-1)!(n-r+1)!}{n!} = \frac{1}{3}$$

By multiplying

$$\Rightarrow \frac{(r-1)\,(r-2)(n-r+1)}{(r-2)!(n-r+2)!} = \frac{1}{3}$$

$$\Rightarrow \frac{(r-1)(n-r+1)!}{(n-r+2)(n-r+1)!} = \frac{1}{3}$$

On simplifying we get

$$\Rightarrow \frac{(r-1)}{(n-r+2)} = \frac{1}{3}$$

⇒ 3r – 3 = n – r + 2

⇒ n – 4r + 5 = 0 …………1

Also

$$\frac{\text{the coeffiecient of } r^{th} \text{ term}}{\text{cofficient of } r+1^{th} \text{ term}} = \frac{1}{3}$$

$$\Rightarrow \frac{\frac{n!}{(r-1)!\,(n-r+1)!}}{\frac{n!}{r!\,(n-r)!}} = \frac{1}{3}$$

On rearranging we get

$$\Rightarrow \frac{n!}{(r-1)!(n-r+1)!} \times \frac{r!(n-r)!}{n!} = \frac{1}{3}$$

By multiplying

$$\Rightarrow \frac{r\,(r-1)(n-r)!}{(r-1)!(n-r+1)!} = \frac{1}{3}$$

$$\Rightarrow \frac{r(n-r)!}{(n-r+1)!} = \frac{1}{3}$$

$$\Rightarrow \frac{r(n-r)!}{(n-r+2)(n-r)!} = \frac{1}{3}$$

On simplifying we get

$$\Rightarrow \frac{r(}{(n-r+1)} = \frac{1}{3}$$

$$\frac{\text{the coeffiecient of } r^{th} \text{ term}}{\text{cofficient of } r+1^{th} \text{ term}} = \frac{1}{3}$$

$$\Rightarrow \frac{\frac{n!}{(r-1)!\,(n-r+1)!}}{\frac{n!}{r!\,(n-r)!}} = \frac{1}{3}$$

On rearranging we get

⇒ 5r = 3n – 3r + 3

⇒ 8r – 3n - 3 =0………….2

We have 1 and 2 as

n - 4r ± 5 =0…………1

$8r - 3n - 3 = 0$……………..2
Multiplying equation 1 by number 2
$2n - 8r + 10 = 0$……………….3
Adding equation 2 and 3
$2n - 8r + 10 = 0$
$-3n - 8r - 3 = 0$
$\Rightarrow -n = -7$
$n = 7$ and $r = 3$

**Question 11**

**Prove that the coefficient of xn in the expansion of $(1 + x)^{2n}$ is twice the coefficient of $x^n$ in the expansion of $(1 + x)2^{n-1.}$**

**Answer:**

The general term $T_{r+1}$ in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r} b^r$
The general term for binomial $(1+x)^{2n}$ is
$T_{r+1} = {}^{2n}C_r\, x^r$ …………………..1
To find the coefficient of $x^n$
$r = n$
$T_{n+1} = {}^{2n}C_n\, x^n$
The coefficient of $x^n = {}^{2n}C_n$
The general term for binomial $(1+x)^{2n-1}$ is
$T_{r+1} = {}^{2n-1}C_r\, x^r$
To find the coefficient of $x^n$
Putting $n = r$
$T_{r+1} = {}^{2n-1}C_r\, x^n$
The coefficient of $x^n = {}^{2n-1}C_n$
We have to prove
Coefficient of $x^n$ in $(1+x)^{2n} = 2$ coefficient of $x^n$ in $(1+x)^{2n-1}$
Consider LHS $= {}^{2n}C_n$
$= \frac{2n!}{n\,!(2n!-n)!}$
$= \frac{2n!}{n!(n)!}$
Again consider RHS $= 2 \times {}^{2n-1}C_n$
$= 2 \times \frac{(2n-1)!}{n!(2n-1-n)!}$
$= 2 \times \frac{(2n-1)!}{n!(n-1)!}$
Now multiplying and dividing by n we get
$= 2 \times \frac{(2n-1)!}{n!(n-1)!} \times \frac{n}{n}$
$= \frac{2n\,(2n-1)!}{n!n\,(n-1)!}$

$= \frac{2n!}{n!n!}$

From above equations LHS = RHS

Hence proof

**Question 12**

**Find a positive value of m for which the coefficient of $x^2$ in the expansion $(1 + x)^m$ is 6.**

**Answer:**

The general term $T_{r+1}$ in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Here a = 1, b = x and n = m

Putting the value

$T_{r+1} = {}^m C_r\, 1^{m-r}\, x^r$

$= {}^m C_r\, x^r$

We need coefficient of $x^2$

∴ putting r = 2

$T_{2+1} = {}^mC_2\, x^2$

The coefficient of $x^2 = {}^mC_2$

Given that coefficient of $x^2 = {}^mC_2 = 6$

$\Rightarrow \frac{m!}{2!(m-2)!} = 6$

$\Rightarrow \frac{m\ (m-1)(m-2)!}{2 \times 1 \times (m-2)!} = 6$

$\Rightarrow m\ (m - 1) = 12$

$\Rightarrow m^2 - m - 12 = 0$

$\Rightarrow m^2 - 4m + 3m - 12 = 0$

$\Rightarrow m\ (m - 4) + 3\ (m - 4) = 0$

$\Rightarrow (m+3)\ (m - 4) = 0$

$\Rightarrow m = -3, 4$

We need positive value of m so m = 4

# Miscellaneous Exercise

**Question 1**

**Find a, b and n in the expansion of $(a + b)^n$ if the first three terms of the expansion are 729, 7290 and 30375, respectively.**

**Answer:**

We know that $(r + 1)^{th}$ term, $(T_{r+1})$, in the binomial expansion of (a + b) n is given by

$T_{r+1} = {}^nC_r\, a^{n-t}\, b^r$

The first three terms of the expansion are given as 729, 7290 and 30375 respectively.
Then we have,
$T_1 = {}^nC_0\, a^{n-0}\, b^0 = a^n = 729$….. 1
$T_2 = {}^nC_1\, a^{n-1}\, b^1 = na^{n-1}\, b = 7290$…. 2
$T_3 = {}^nC_2\, a^{n-2}\, b^2 = n\,(n-1)/2$ an-2 b2 = 30375……3
Dividing 2 by 1 we get
$na^{n-1}\, ba^n = \frac{7290}{729}$
n ba = 10 ………….4
Dividing 3 by 2 we get
$n\,(n-1)\, a^{n-2}\, b^2\, 2na^{\, n-1} b\, \frac{30375}{7290}$
$\Rightarrow (n-1)\, b2a = \frac{30375}{7290}$
$\Rightarrow (n-1)\, ba = \frac{30375 \times 2}{7290} = \frac{25}{3}$
$\Rightarrow nba - \frac{b}{a} = \frac{25}{3}$
$\Rightarrow 10 - ba = \frac{25}{3}$
$\Rightarrow ba = 10 - \frac{25}{3} = \frac{5}{3}$ ………..(5)
From 4 and 5 we have
n. 5/3 = 10
n = 6
Substituting n = 6 in 1 we get
$a^6 = 729$
a = 3
From 5 we have, b/3 = 5/3
b = 5
Thus a = 3, b = 5 and n = 76

## Question 2

**Find a if the coefficients of x2 and x3 in the expansion of $(3 + a\,x)^9$ are equal.**

**Answer:**

We know that general term of expansion $(a + b)^n$ is
$T_{r+1} = \binom{n}{r} a^{n-r} b^r$
For $(3 + ax)^9$
Putting a = 3, b = a × & n = 9
General term of $(3 + ax)^9$ is
$T_{r+1} = \binom{9}{r} 3^{n-r} (ax)^r$
$T_{r+1} = \binom{9}{r} 3^{n-r} a^r x^r$
Since we need to find the coefficient of $x^r$ and $x^3$ therefore

For r = 2

$T_{r+1} = \binom{9}{3} 3^{n-3} a^3 x^3$

Thus coefficient of $x^3$ $\binom{9}{3} 3^{n-3} a^3$

Given that coefficient of $x^2$ = coefficient of $x^3$

$\Rightarrow \binom{9}{2} 3^{n-2} a^2 = \binom{9}{3} 3^{n-3} a^3$

$\Rightarrow \frac{9!}{2!(9-2)!} \times 3^{n-2} a^2 = \frac{9!}{3!(9-3)!} \times 3^{n-3} a^3$

$\Rightarrow \frac{3^{n-2} a^2}{3^{n-3} a^3} = \frac{2!(9-2)!}{3!(9-3)!}$

$\Rightarrow \frac{3^{(n-2)-(n-3)}}{a} = \frac{2!7!}{3!6!}$

$\Rightarrow \frac{3}{a} = \frac{7}{3}$

$\therefore a = 9/7$

Hence, a = 9/7

## Question 3

**Find the coefficient of x5 in the product (1 + 2x)6 (1 – x)7 using binomial theorem.**

**Answer:**

$(1 + 2x)^6 = {}^6C_0 + {}^6C_1 (2x) + {}^6C_2 (2x)^2 + {}^6C_3 (2x)^3 + {}^6C_4 (2x)^4 + {}^6C_5 (2x)^5 + {}^6C_6 (2x)^6$

$= 1 + 6 (2x) + 15 (2x)^2 + 20 (2x)^3 + 15 (2x)^4 + 6 (2x)^5 + (2x)^6$

$= 1 + 12 x + 60x^2 + 160 x^3 + 240 x^4 + 192 x^5 + 64x^6$

$(1 – x)^7 = {}^7C_0 - {}^7C_1 (x) + {}^7C_2 (x)^2 - {}^7C_3 (x)^3 + {}^7C_4 (x)^4 - {}^7C_5 (x)^5 + {}^7C_6 (x)^6 - {}^7C_7 (x)^7$

$= 1 – 7x + 21x^2 – 35x^3 + 35x^4 – 21x^5 + 7x^6 – x^7$

$(1 + 2x)^6 (1 – x)^7 = (1 + 12 x + 60x^2 + 160 x^3 + 240 x^4 + 192 x^5 + 64x^6) (1 – 7x + 21x^2 – 35x^3 + 35x^4 – 21x^5 + 7x^6 – x^7)$

192 – 21 = 171

Thus, the coefficient of x5 in the expression $(1+2x)^6(1-x)^7$ is 171.

## Question 4

**If a and b are distinct integers, prove that a – b is a factor of an – bn, whenever n is a positive integer. [Hint write $a^n = (a – b + b)^n$ and expand]**

**Answer:**

In order to prove that (a – b) is a factor of $(a^n – b^n)$, it has to be proved that

$a^n - b^n = k (a – b)$ where k is some natural number.

a can be written as a = a – b + b

$a^n = (a – b + b)^n = [(a – b) + b]^n$

$= {}^nC_0 (a – b)^n + {}^nC_1 (a – b)^{n-1} b + \ldots\ldots + {}^nC_n b^n$

$a^n - b^n = (a - b)\,[(a - b)^{n-1} + {}^nC_1\,(a - b)^{n-1}\,b + \ldots\ldots + {}^nC_n\,b^n]$
$a^n - bn = (a - b)\,k$
Where $k = [(a - b)^{n-1} + {}^nC_1\,(a - b)^{n-1}\,b + \ldots\ldots + {}^nC_n\,b^n]$ is a natural number
This shows that $(a - b)$ is a factor of $(a^n - b^n)$, where n is positive integer.

**Question 5**

**Evaluate**
$\left(\sqrt{3} + \sqrt{2}\right)^6 - \left(\sqrt{3} - \sqrt{2}\right)^6$

**Answer:**

Using binomial theorem the expression $(a + b)^6$ and $(a - b)^6$, can be expanded
$(a + b)^6 = {}^6C_0\,a^6 + {}^6C_1\,a^5\,b + {}^6C_2\,a^4\,b^2 + {}^6C_3\,a^3\,b^3 + {}^6C_4\,a^2\,b^4 + {}^6C_5\,a\,b^5 + {}^6C_6\,b^6$
$(a - b)^6 = {}^6C_0\,a^6 - {}^6C_1\,a^5\,b + {}^6C_2\,a^4\,b^2 - {}^6C_3\,a^3\,b^3 + {}^6C_4\,a^2\,b^4 - {}^6C_5\,a\,b^5 + {}^6C_6\,b^6$
Now $(a + b)^6 - (a - b)^6 = {}^6C_0\,a^6 + {}^6C_1\,a^5\,b + {}^6C_2\,a^4\,b^2 + {}^6C_3\,a^3\,b^3 + {}^6C_4\,a^2\,b^4 + {}^6C_5\,a\,b^5 + {}^6C_6\,b^6$
$- [{}^6C_0\,a^6 - {}^6C_1\,a^5\,b + {}^6C_2\,a^4\,b^2 - {}^6C_3\,a^3\,b^3 + {}^6C_4\,a^2\,b^4 - {}^6C_5\,a\,b^5 + {}^6C_6\,b^6]$
Now by substituting $a = \sqrt{3}$ and $b = \sqrt{2}$ we get
$(\sqrt{3} + \sqrt{2})^6 - (\sqrt{3} - \sqrt{2})^6 = 2\,[6\,(\sqrt{3})^5\,(\sqrt{2}) + 20\,(\sqrt{3})^3\,(\sqrt{2})^3 + 6\,(\sqrt{3})\,(\sqrt{2})^5]$
$= 2\,[54(\sqrt{6}) + 120\,(\sqrt{6}) + 24\,\sqrt{6}]$
$= 2\,(\sqrt{6})\,(198)$
$= 396\,\sqrt{6}$

**Question 6**

**Find the value of**
$\left(\mathbf{a^2} + \sqrt{\mathbf{a^2 - 1}}\right)^4 - \left(\mathbf{a^2} - \sqrt{\mathbf{a^2 - 1}}\right)^4$

**Answer:**

Firstly the expression $(x + y)^4 + (x - y)^4$ is simplified by using binomial theorem
$(x + y)^4 = {}^4C_{0x}{}^4 + {}^4C_1x^3y + {}^4C_2x^2y^2 + {}^+C_{3x}y3 + {}^4C_{4y}{}^4$
$= x^4 + 4_x{}^3y + 6x^2y^2 + 4_xy^3 + y^4$
$(x - y)^4 = {}^4C_{0x}{}^4 - {}^4C_1x^3y - {}^4C_2x^2y^2 - {}^4C_3xy3 + {}^4C_{4y}{}^4$
$= x^4 + 4_x{}^3y + 6x^2y^2 - 4_xy^3 + y^4$
$\therefore (x + y)^4 + (x - y)^4 = 2\,(x^4 + 6x^2y^2 + y^4)$
Putting $x = a^2$ and $y = \sqrt{a^2 - 1}$, we obtain
$\left(a^2 + \sqrt{a^2 - 1}\right)^4 - \left(a^2 - \sqrt{a^2 - 1}\right)^4$
$= 2\left[(a^2)^4 + 6(a^2)^2\left(\sqrt{a^2 - 1}\right)^2 + \left(\sqrt{a^2 - 1}\right)^4\right]$
$= 2\,[a^8 + 6a^4\,(a^2 - 1) + (a^2 - 1)^2]$
$= 2\,[a^8 + 6a^6 - 6a^4 + a^4 - 2a^2 + 1]$

$= 2\ [a^8 + 6a^6 - 5\,a^4 - 2a^2 + 1]$
$= 2a^8 + 12a^6 - 10a^4 - 4a^2 + 2$

## Question 7

**Find an approximation of (0.99)5 using the first three terms of its expansion.**

**Answer:**

0.99 can be written as
0.99 = 1 – 0.01
Now by applying binomial theorem we get
$(o.\,99)^5 = (1 - 0.01)^5$
$= {}^5C_0\,(1)^5 - {}^5C_1\,(1)^4\,(0.01) + {}^5C_2\,(1)^3\,(0.01)^2$
$= 1 - 5\,(0.01) + 10\,(0.01)^2$
$= 1 - 0.05 + 0.001$
$= 0.951$

## Question 8

**Find n, if the ratio of the fifth term from the beginning to the fifth term from the end in the expansion $\left(\sqrt[4]{2} + \frac{1}{\sqrt[4]{3}}\right)^n$ of is $\sqrt{6}: 1$**

**Answer:**

In the expansions $(a + b)^4$, if n is even then the middle term is $(n/2 + 1)^{th}$ term

${}^nC_a\left(\sqrt[4]{2}\right)^{n-1}\ \left(\frac{1}{\sqrt[4]{3}}\right)^4 = {}^nC_4\frac{\left(\sqrt[4]{2}\right)^n}{\left(\sqrt[4]{2}\right)^4}\cdot\frac{1}{3} = {}^nC_4\,\frac{\left(\sqrt[4]{2}\right)^n}{2}\cdot\frac{1}{3} = \frac{n!}{6.4!(n-4)!}\left(\sqrt[4]{2}\right)^n$

${}^nC_{n-4}\left(\sqrt[4]{2}\right)^4\ \left(\frac{1}{\sqrt[4]{3}}\right)^{n-4} = {}^nC_{n-1}\,.\,2\,.\,\frac{\left(\sqrt[4]{2}\right)^n}{\left(\sqrt[4]{2}\right)^4}\,.\,{}^nC_4\,.\,2.\,\frac{3}{\left(\sqrt[4]{3}\right)^n} = \frac{6n!}{(n-4)4!}\cdot\frac{1}{\left(\sqrt[4]{3}\right)^n}$

$\frac{6n!}{6.4(n-4)4!}\cdot\left(\sqrt[4]{3}\right)^n : \frac{6n!}{(n-4)4!}\ \frac{1}{\left(\sqrt[4]{3}\right)^n} = \sqrt{6} : 1$

$\Rightarrow \frac{\left(\sqrt[4]{2}\right)^n}{6} : \frac{6}{\left(\sqrt[4]{3}\right)^n} = \sqrt{6} : 1$

$\Rightarrow \frac{\left(\sqrt[4]{2}\right)^n}{6} \times \frac{\left(\sqrt[4]{3}\right)^n}{6} = \sqrt{6}$

$\Rightarrow\left(\sqrt[4]{6}\right)^n = 36\,\sqrt{6}$

$\Rightarrow 6^{\frac{n}{4}} = 6^{\frac{5}{2}}$

$\Rightarrow\frac{n}{4} = \frac{5}{2}$

$\Rightarrow n = 4 \times \frac{5}{2} = 10$

Thus the value of n = 10

## Question 9

**Expand using Binomial Theorem**

$\left(1+\frac{x}{2}-\frac{2}{x}\right)^4, x \neq 0$

**Answer:**

$\left(1+\frac{x}{2}-\frac{2}{x}\right)^4, x \neq 0$

$= {}^4C_0\left(1+\frac{x}{2}\right)^4 - {}^4C_1\left(1+\frac{x}{2}\right)^3\left(\frac{2}{x}\right) + {}^4C_2\left(1+\frac{x}{2}\right)^2\left(\frac{2}{x}\right)^2 - {}^4C_3\left(1+\frac{x}{2}\right)\left(\frac{2}{x}\right)^3 + {}^4C_4\left(\frac{2}{x}\right)^4$

$= \left(1+\frac{x}{2}\right)^4 - 4\left(1+\frac{x}{2}\right)^3\left(\frac{2}{x}\right) + 6\left(1+x+\frac{x^2}{4}\right)\left(\frac{4}{x^2}\right) - 4\left(1+\frac{x}{2}\right)\left(\frac{8}{x^3}\right) + \frac{16}{x^4}$

$= \left(1+\frac{x}{2}\right)^4 - \frac{8}{x}\left(1+\frac{x}{2}\right)^3 + \frac{24}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} - \frac{16}{x^2} + \frac{16}{x^4}$

$= \left(1+\frac{x}{2}\right)^4 - \frac{8}{x}\left(1+\frac{x}{2}\right)^3 + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4}$ ………… (1)

Again by using binomial theorem to expand the above terms we get

$= \left(1+\frac{x}{2}\right)^4 = {}^4C_0 - (1)^4 + {}^4C_1(1)^3\left(\frac{x}{2}\right){}^4C_2(1)^1\left(\frac{x}{2}\right)^3 + {}^4C_4\left(\frac{x}{2}\right)^4$

$= 1 + 4\times\frac{x}{2} + 6\times\frac{x^2}{4} + 4\times\frac{x^3}{8} + \frac{x^4}{16}$

$= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16}$ ………… (2)

$= \left(1+\frac{x}{2}\right)^3 = {}^3C_0 - (1)^3 + {}^3C_1(1)^3\left(\frac{x}{2}\right) + {}^3C_2(1)\left(\frac{x}{2}\right)^3 + {}^3C_3\left(\frac{x}{2}\right)^3$

$= 1 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{x^3}{8}$ ………….. (3)

From equation 1, 2 and 3 we get

$= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - \frac{8}{x}\left(1+\frac{3x}{2}+\frac{3x^2}{4}+\frac{x^3}{8}\right) + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4}$

$= 1 + 2x + \frac{3}{2}x^2 + \frac{x^3}{2} + \frac{x^4}{16} - \frac{8}{x} - 12 - 6x - x^2 + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4}$

$= \frac{16}{x} + \frac{8}{x^2} - \frac{32}{x^3} + \frac{16}{x^4} - 4x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - 5$

## Question 10

**Find the expansion of $(3x2 - 2ax + 3a^2)^3$ using binomial theorem.**

**Answer:**

We know that $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Putting $a = 3x^2$ & $b = -a(2x-3a)$, we get

$[3x^2 + (-a(2x-3a))]^3$

$= (3x2)^3+3(3x^2)^2(-a(2x-3a)) + 3(3x^2)(-a(2x-3a))^2 + (-a(2x-3a))^3$

$= 27x^6 - 27ax^4 (2x-3a) + 9a^2x^2 (2x-3a)^2 - a^3 (2x-3a)^3$
$= 27x^6 - 54ax^5 + 81a^2x^4 + 9a^2x^2 (4x^2-12ax+9a^2) - a^3 [(2x)^3 - (3a)^3 - 3(2x)^2(3a) + 3(2x)(3a)^2]$
$= 27x^6 - 54ax^5 + 81a^2x^4 + 36a^2x^4 - 108a^3x^3 + 81a^4x^2 - 8a^3x^3 + 27a^6 + 36a^4x^2 - 54a^5x$
$= 27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$
Thus, $(3x^2 – 2ax + 3a^2)^3$
$= 27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$