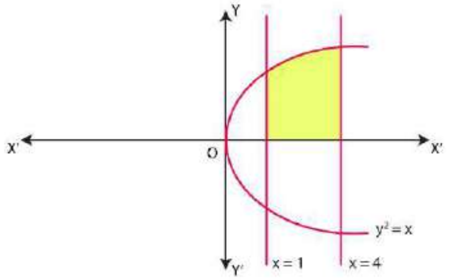
Chapter 8
Application of Integrals
Exercise 8.1

Question 1

Find the area of region bounded by the curve $y^2 = x$ and the x = 1, x = 4 then x – axis in the first quadrant

Solution:

Equation of the curve (rightward parabola) is $y^2 = x$



$y = \sqrt{x}$ ………… (1)
Required area is shaded region

$= \left|\int_1^4 y\,dx\right| = \left|\int_1^4 \sqrt{x}\,dx\right|$ [From equation (1)]

$= \left|\int_1^4 x^{\frac{1}{2}}\,dx\right|$

$= \left|\frac{\left(x^{\frac{3}{2}}\right)_1^4}{\frac{3}{2}}\right|$

$= \left|\frac{2}{3}\left(4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right)\right|$

$= \left|\frac{2}{3}\left(4^{\frac{3}{2}\times 3} - 1^{\frac{3}{2}\times 3}\right)\right| = \left|\frac{2}{3}(8-1)\right| = \frac{2}{3}\times 7 = \frac{14}{3}$ sq. units

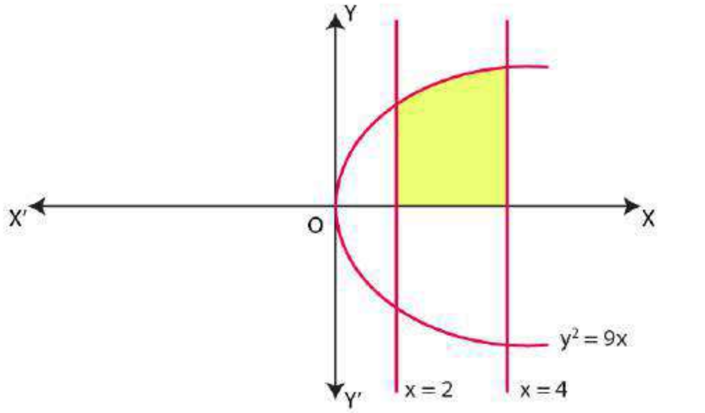
Question 2

Find the area of the region bounded by $y^2 = 9x, x = 2, x = 4$ and the a-axis in the first quadrant.

Solution:

Equation of the curve (rightward parabola) is$y^2 = 9x$.

$y = 3\sqrt{x}$



Required area is shaded region which is bounded by curve $y^2 = 9x$ and vertical lines x=2,x=4 and x-axis in first quadrant.

$= \left|\int_1^4 y\, dx\right| = \left|\int_1^4 3\sqrt{x}\, dx\right|$ [From equation (1)]

$= \left|3\int_1^4 x^{\frac{1}{2}}\, dx\right| = \left|3\frac{\left(x^{\frac{3}{2}}\right)_2^4}{\frac{3}{2}}\right|$

$= \left|3.\frac{2}{3}\left(4^{\frac{3}{2}} - 2^{\frac{3}{2}}\right)\right| = \left|3.\frac{2}{3}\left(4^{\frac{1}{2}\times 3} - 2^{\frac{1}{2}\times 3}\right)\right|$

$= \left|2\,(8 - 2\sqrt{2})\right| = (16 - 4\sqrt{2})$ sq. unit.

Question 3

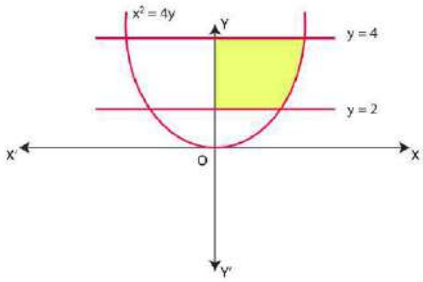
Find the area of the region bounded by $x^2 = 4y. y = 2. y = 4$ and the y – axis in the first quadrant.

Solution:

Equation of curve (parabola) is $x^2 = 4y$.

Or $x = 2\sqrt{y}$ .... (1)

Required region is shaded, that is area bounded by curve$x^2 = 4y$. and Horizontal lines . $y = 2. y = 4$ and y-axis in first quadrant.



$= \left|\int_2^4 x\,dy\right| = \left|\int_2^4 2\sqrt{y}\,dy\right| = \left|2\int_2^4 y^{\frac{1}{2}}\,dy\right|$

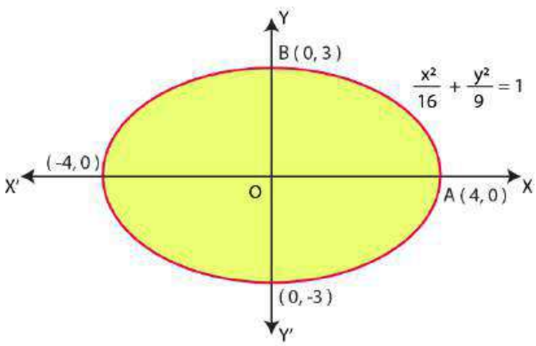
$= \left|2\frac{\left(y^{\frac{3}{2}}\right)_2^4}{\frac{3}{2}}\right| = \frac{4}{3}\left(4^{\frac{3}{2}} - 2^{\frac{3}{2}}\right) = \left(\frac{32-8\sqrt{2}}{3}\right)$ sq. units

Question 4

Find the area of the region bounded by the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

Solution:

Equation of ellipse is $\frac{x^2}{16} + \frac{y^2}{9} = 1$ ………… (1)



Here $a^2\ (= 6) > \ b^2\ (= 9)$
From equation (1), $\frac{y^2}{9} + \frac{x^2}{16} = \frac{16-x^2}{16}$
$\Rightarrow y^2 = \frac{9}{16}(16 - \ x^2)$
$\Rightarrow y^2 = \frac{3}{4}(16 - \ x^2)$ ………. (2)
For arc of ellipse in first quadrant.
Ellipse (1) is symmetrical about x-axis (if we change y to – y or x to – x, equation remain same).
Intersections of ellipse 91) with x-axis (y=0)

Put y = 0 in equation (1), we have

$\frac{x^2}{16} = 1 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$

Therefore, Intersection of ellipse (1) with x-axis are (0,4) and (0,4)

Now again

Intersection of ellipse (1) with y-axis (x=0)

Putting x = 0 in equation (1), $\frac{y^2}{9} = 1 \Rightarrow y^2 = 9 \Rightarrow y = \pm 3$

Therefore, Intersection of ellipse (1) with y-axis are (0, 3) and (0,-3).

Now,

Area of region bounded by ellipse (1) = Total shaded are = 4 × Area OAB of ellipse in first quadrant.

$= 4 \left| \int_0^4 y\, dx \right|$ [∵ At end B arc AB of ellipse; x = 0 and at end A of arc AB; x −= 4]

$= 4 \left| \int_0^4 \frac{3}{4}\sqrt{16 - x^2}\, dx \right| = 4 \left| \int_0^4 \frac{3}{4}\sqrt{4^2 - x^2}\, dx \right|$

$= 3 \left[\frac{x}{2}\sqrt{4^2 - x^2} + \frac{4^2}{2} sin^{-1}\frac{x}{4}\right]_0^4$ $\left[\because \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} sin^{-1}\frac{x}{a}\right]$

$= 3\left[\frac{4}{2}\sqrt{16 - 16} + 8\, sin^{-1}1 - (0 + 8\, sin^{-1}0)\right] = 3\left[0 + \frac{8\pi}{2}\right]$

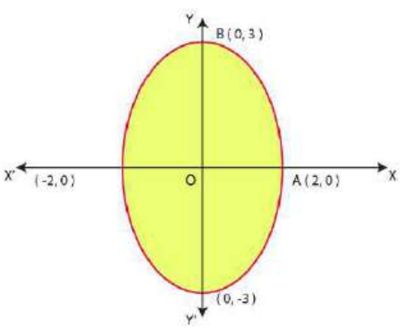
$= 3\,(4\pi) = 12\pi$ sq. units

Question 5

Find the area of the region bounded by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

Solution:

Equation of ellipse is $\frac{x^2}{4} + \frac{y^2}{9} = 1$



Here $a^2\ (= 4) < b^2\ (= 9)$

From equation (1), $\frac{y^2}{9} = 1 - \frac{x^2}{4} = \frac{4 - x^2}{4}$

$\Rightarrow y^2 = \frac{9}{4}\,(4 - x^2)$

$\Rightarrow y^2 = \frac{3}{2}(4 - x^2)$ ………… (2)

For an arc of ellipse in first quadrant.

Ellipse (1) is symmetrical about x-axis and y-axis.

Intersections of ellipse (1) with x-axis (y=0)

Put y=0 in equation (1), $\frac{x^2}{4} = 1$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$

Therefore, Intersections of ellipse (1) with x-axis are (0, 2) and (0,-2).

Intersections of ellipse (1) with y-axis are $(\times = 0)$

Put $x = 0$ in equation (1), $\frac{y^2}{9} = 1$

$\Rightarrow y^2 = 9 \Rightarrow y = \pm 3$

Therefore, Intersection of ellipse (1) with y-axis (0,3) and (0,-3)

Now

Area of region bounded by ellipse (1) = Total shaded area = 4 × Area OAB of ellipse in first quadrant

$= 4\left|\int_0^2 y\,dx\right|$ [∵ At end B arc AB of ellipse; x = 0 and at end A of arc AB; x = 2]

$= 4\left|\int_0^2 \frac{3}{4}\sqrt{4 - x^2}\,dx\right| = 4\left|\int_0^4 \frac{3}{2}\sqrt{2^2 - x^2}\,dx\right|$

$= 6\left[\frac{x}{2}\sqrt{2^2 - x^2} + \frac{2^2}{2}\sin^{-1}\frac{x}{4}\right]_0^4$ $\left[\because \int\sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$

$= 6\left[\frac{2}{2}\sqrt{4 - 4} + 2\sin^{-1}1 - (0 + 2\sin^{-1}0)\right]$

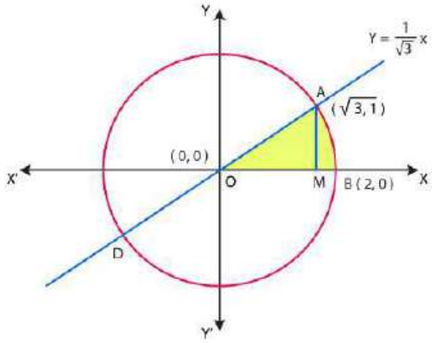
$= 6\left[0 + 2.\frac{\pi}{2} - 0\right] = 6\pi$ sq.unit

Question 6

Find the area of the region in the first quadrant enclosed by x – axis $x = \sqrt{3}y$ and the circle $x^2 + y^2 = 4$

Solution:

Step 1: To draw the graphs and shade the region whose we are to find



Equation of the circle is $x^2 + y^2 = 2^2$ ………. (1)
We know that equation (1) represents a circle whose centre is (0,0) and radius is 2
Equation of given line is $x = \sqrt{3}y$
$\Rightarrow$ y = $\frac{1}{\sqrt{3}}x$ ………… (2)
We know that equation (2) being of the from $y = mx$ where $m = \frac{1}{\sqrt{3}}$= tan 30^0 = tan$\theta \Rightarrow \theta = 30^0$
represents a straight line passing through the origin and making angle of 30^0 with x- axis
Step 2: To find the value of x and y
Put y = $\frac{1}{\sqrt{3}}$ from equation (2) in equation (1),
$x^2 + \frac{x^2}{3} = 4 \Rightarrow 3x^2 + x^2 = 12 \Rightarrow 4x^2 = 12$
$\Rightarrow x^2 = 3 \Rightarrow x = \pm 3$
Putting $x = \pm 3$ in $y = \frac{1}{\sqrt{3}}$, y = 1 and y = -1
Therefore the two points of intersections of circle (1) and line (2) are A $\left(\sqrt{3}, 1\right)$ and D $\left(-\sqrt{3}, -1\right)$
Step 3: Now shaded area OAM between segment OA of line (2) and x –axis

$$= \left|\int_0^{\sqrt{3}} y\,dx\right| [\because \text{ At O, x = 0 and at A, x } \sqrt{3}]$$

$$= \left|\int_0^{\sqrt{3}} \frac{1}{\sqrt{3}}\,x\,dx\right| = \frac{1}{\sqrt{3}}\left(\frac{x^2}{2}\right)_0^{\sqrt{3}} = \frac{1}{\sqrt{3}}\left(\frac{3}{2} - 0\right) = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2} \text{ sq. units } \ldots\ldots\ldots\ldots\ldots (3)$$

Step IV: Now shaded area AMB between are AB of circle and x – axis.

$$= \left|\int_{\sqrt{3}}^{2} y\,dx\right| [\because \text{ At O, x = } \sqrt{3} \text{ and at A, x = 2}]$$

$$= \left|\int_{\sqrt{3}}^{2} \sqrt{2^2 - x^2}\,dx\right| \text{from equation(2),}$$

$\left(\frac{x}{2}\sqrt{2^2 - x^2} + \frac{2^2}{2}\, sin^{-1}\frac{x}{2}\right)_{\sqrt{3}}^{2} = \left[\frac{2}{2} - \sqrt{4-4} + 2\, sin^{-1}1 - \left(\frac{\sqrt{3}}{2}\sqrt{4-3} + 2\, sin^{-1}\frac{\sqrt{3}}{2}\right)\right]$

$= 0 + 2.\frac{\pi}{2} - \frac{\sqrt{3}}{2} - 2.\frac{\pi}{3} = \pi - \frac{\sqrt{3}}{2} - \frac{2\pi}{3} = \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$ sq. units ………. (iv)

Step V: Required shaded area OAB = Area of OAM + Area of AMB

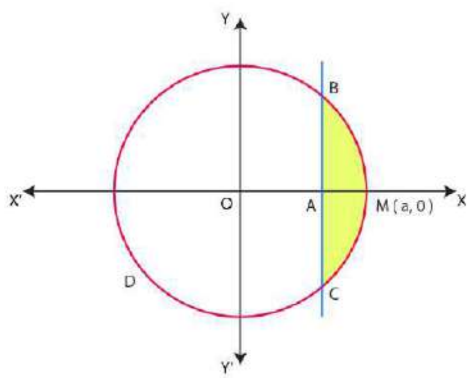
$= \frac{\sqrt{3}}{2} + \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ sq. units

Question 7

Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off the line $x = \frac{a}{\sqrt{2}}$

Solution:

Equation of the circle is $x^2 + y^2 = a^2$ …………… (1)



$\therefore y^2 = a^2 - x^2$

$\Rightarrow$ y = $\sqrt{a^2 - x^2}$ …………… (1)

Here,

Area of smaller part of the circle $x^2 + y^2 = a^2$ cut off the line $x = \frac{a}{\sqrt{2}}$ = Area of ABMC = 2 × Area of ABM

$$= 2\left|\int_{\frac{a}{\sqrt{2}}}^{a} y\,dx\right| = 4\left|\int_{\frac{a}{\sqrt{2}}}^{a}\frac{3}{2}\sqrt{a^2 - x^2}\,dx\right| \text{[From equation (2)]}$$

$$= 2\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\, sin^{-1}\frac{x}{a}\right]_{\frac{a}{\sqrt{2}}}^{a}$$

$$=2\left[\frac{a}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\, sin^{-1} 1 - \left(\frac{\frac{a^2}{\sqrt{2}}}{2}\sqrt{a^2 - \frac{a^2}{2}}\, sin^{-1}\frac{\frac{a^2}{\sqrt{2}}}{2}\right)\right]$$

$= 2\left[0 + \frac{a^2}{2}.\frac{\pi}{2} - \frac{a}{2\sqrt{2}}\sqrt{\frac{a^2}{2}} - \frac{a^2}{2}\, sin^{-1}\frac{1}{\sqrt{2}}\right]$

$= 2\left[\frac{\pi\, a^2}{4} - \frac{a}{2\sqrt{2}}\,\frac{a}{\sqrt{2}} - \frac{a^2}{2}\,\frac{\pi}{4}\right]$

$= 2\left[\frac{\pi\, a^2}{4} - \frac{\pi a^2}{8} - \frac{a^2}{4}\right]$

$= 2a^2\left[\frac{2\,\pi - \pi - 2}{8}\right]$

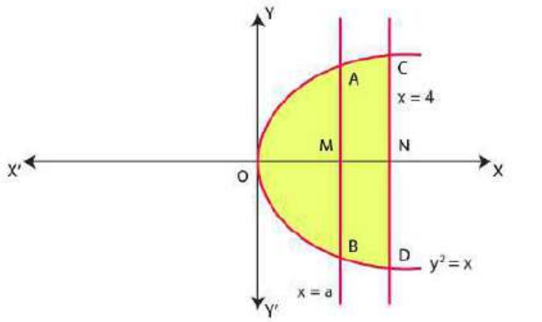
$= \frac{a^2}{4}\,(\pi - 2) = \frac{a^2}{4}\left(\frac{\pi}{2} - 1\right)$ sq unit

Question 8

The area between x = y^2 and x = 4 is divided into equal parts by the line x = a find the value of a.

Solution:

Equation of the curve (parabola) is x = y^2 .... (1)



$\Rightarrow$ y = $\sqrt{x}$

Now area bounded by parabola (1) and vertical line x = 4 is divided into two equal parts vertical line x = a.

Area OAMB = Area AMBDNC

$\Rightarrow\ 2\left|\int_0^a y\, dx\right| = 2\left|\int_a^4 y\, dx\right|$

$\Rightarrow 2\left|\int_0^a x^{\frac{1}{2}}\, dx\right| = 2\left|\int_a^4 x^{\frac{1}{2}}\, dx\right|$

$\Rightarrow \frac{\left(x^{\frac{3}{2}}\right)_0^a}{\frac{3}{2}} = \frac{\left(x^{\frac{3}{2}}\right)_a^4}{\frac{3}{2}}$

$\Rightarrow \frac{2}{3}\left[a^{\frac{3}{2}} - 0\right] = \frac{2}{3}\left[4^{\frac{3}{2}} - a^{\frac{3}{2}}\right]$

$\Rightarrow a^{\frac{3}{2}} = 8 - a^{\frac{3}{2}}$

$\Rightarrow 2a^{\frac{3}{2}} = 8 \Rightarrow a^{\frac{3}{2}} = 4$

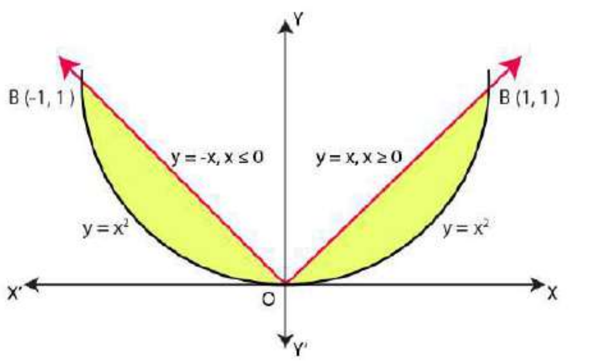
$\Rightarrow a = 4^{\frac{3}{2}}$

Question 9

Find the area of the region bounded by the parabola y = x^2 and y = $|x|$

Solution:

The required area is the area included between the parabola $y = x^2$ and the modulus function y = $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$



To find: Area between the parabola y = x^2 and the ray y = x for x ≥ 0

Here limits of integration ⇒ y = x

$\Rightarrow x^2 = x \Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1) = 0 \Rightarrow x = 0, x = 1$

Now for y =$|x|$, table of values,

$y = x$ if $x \geq 0$

x	0	1	2
y	0	1	2

$y = -x$ if $x \leq 0$

x	0	-1	-2
y	0	1	2

Now Area between y = x^2 and axis – Area between limits x = 0 and x =1

$$= \int_0^1 y\,dx = \int_0^1 x^2 dx = \left(\frac{x^3}{3}\right)_0^1 = \frac{1}{3} \ldots\ldots\ldots\ldots (1)$$

And area of ray y=x and x-axis

$$= \int_0^1 y\,dx = \int_0^1 x\,dx = \left(\frac{x^3}{2}\right)_0^1 = \frac{1}{2} \ldots\ldots\ldots\ldots\ (1)$$

So required shaded area in first quadrant

= Area between ray y=x for x≥ 0 and x-axis – Area between parabola y = x^2 and x-axis in first quadrant

= Area given by equation (2) – Area given by equation (1)

$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ sq. units

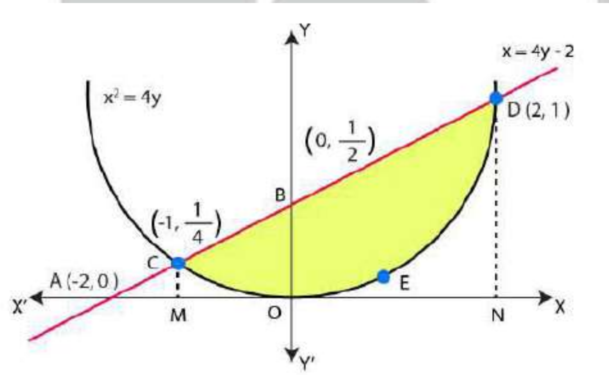
Therefore the required area = 2 × (1/6) = 1/3

Question 10

Find the area bounded by the curve x = 4y and the line x = 4y- 2

Solution:

Step 1: graph and region of integration



Equation of the given curve is

$x^2 = 4y$ ………….. (1)

Equation of the given line is

x = 4y -2………….. (2)

$\Rightarrow y = \frac{x+2}{4}$

x	0	1	-2
y	0	1/2	0

Step 2: putting $y = \frac{x^2}{4}$ from equation (1) in equation (20,

$x = 4\frac{x^2}{4} - 2 \Rightarrow x = x^2 - 2 \Rightarrow -x^2 + x + 2 = 0$

$\Rightarrow x^2 - x - 2 = 0$

$x^2 - 2x + x - 2 = 0 \Rightarrow x(x-2) + (x-2) = 0$

$\Rightarrow (x-2) + (x+1) = 0 \Rightarrow x = 2 \; or \; x = -1$

For x =2, from equation (1), y= $\frac{x^2}{4} = \frac{4}{4} = 1$

So points (2, 1)

For x = 1 from equation (1), y= $\frac{x^2}{4} = \frac{1}{4}$

So point is $\left(-1, \frac{1}{4}\right)$

Therefore, the two points of intersection of parabola (1) and line (2) are C $\left(-1, \frac{1}{4}\right)$ and D (2,1)

Step 3: Area CMOEDN between parabola (1) and x-axis

$$= \left|\int_{-1}^{2} y \, dx\right| = \left|\int_{-1}^{2} \frac{x^2}{4} dx\right|$$

$$= \left|\frac{(x^3)_{-1}^{2}}{12}\right| = \left|\frac{1}{12}\{2^3 + (-1)^3\}\right|$$

$= \frac{1}{12}(8+1) = \frac{9}{12} = \frac{3}{4}$ sq. units……. (3)

Step 4: Area of trapezium CMND between line (2) and x-axis

$$= \left|\int_{-1}^{2} y \, dx\right| = \left|\int_{-1}^{2} \frac{x+2}{4} dx\right|$$

$$= \left|\frac{1}{4}\int_{-1}^{2}(x+2)dx\right| = \frac{1}{4}\left|\left(\frac{x^2}{2} + 2x\right)_{-1}^{2}\right|$$

$= \frac{1}{4}\left|\left(\frac{4}{2} + 4\right) - \left(\frac{1}{2} - 2\right)\right| = \frac{1}{4}\left|\left(2 + 4 - \frac{1}{2} + 2\right)\right|$

$= \frac{1}{4}\left|8 - \frac{1}{2}\right| = \frac{1}{4} \times \frac{15}{2} = \frac{15}{8}$ sq. units……….. (4)

Therefore,

Required shaded area = Area given by equation (4) – Area given by equation (3) = $\frac{15}{8} - \frac{3}{4} = \frac{15-6}{8} = \frac{9}{8}$ sq. units.

Question 11

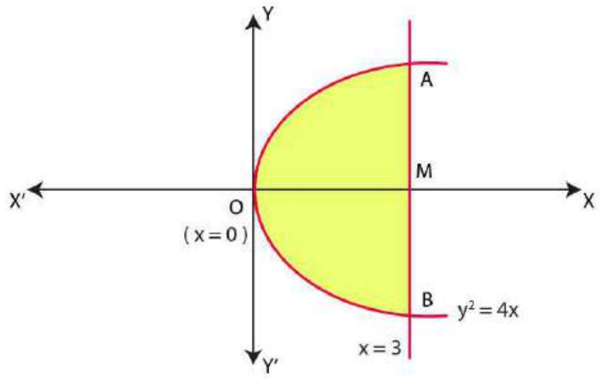
Find the area of the region bounded by the curve $y^2 = 4x$ and the line x = 3

Solution:

$y^2 = 4x$ ………………… (1)

$\Rightarrow y = 4x = 2x^{\frac{1}{2}}$ …………… (2)

Here required shaded area OAMB = 2 × Area OAM



$$= 2\left|\int_0^3 y\,dx\right| = 2\left|\int_0^3 2x^{\frac{1}{2}}dx\right| = 4\left|\frac{\left(x^{\frac{3}{2}}\right)_0^3}{\frac{3}{2}}\right|$$

$$= 4\,.\frac{2}{3}\left[3^{\frac{3}{2}} - 0\right] = \frac{8}{3}.3\sqrt{3} = 8\sqrt{3} \text{ sq. units.}$$

Question 12

Choose the correct answer:

Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the line 0 and x = 2 is

(A) π **(B) $\frac{\pi}{2}$**

(C) $\frac{\pi}{3}$ **(D) $\frac{\pi}{4}$**

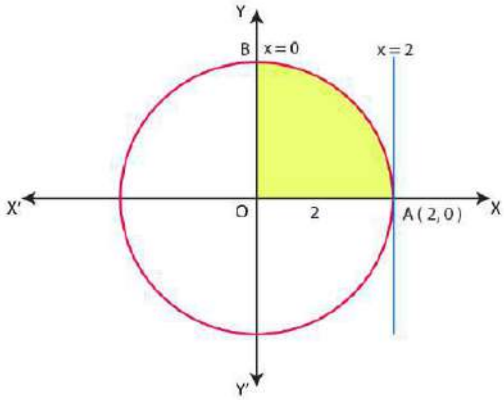
Solution:

Option (A) is correct.

Explanation:

Equation of the circle is $x^2 + y^2 = 2^2$ ………….. (1)

$\Rightarrow y = \sqrt{2^2 - x^2}$ ……………. (2)



Required area $= \left|\int_0^2 y\,dx\right| = \left|\int_0^2 \sqrt{2^2-x^2}\,dx\right|$

$= \left|\left(\frac{x}{2}\sqrt{2^2-x^2} + \frac{2^2}{2}\sin^{-1}\frac{x}{2}\right)_0^2\right|$

$= \frac{2}{2}\sqrt{4-4} + 2\sin^{-1}1 - (0 + 2\sin^{-1}0)$

$= 0 + 2.\frac{\pi}{2} - 0 - 0 = \pi$ sq. units.

Question 13

Choose the correct answer:

Area of the region bounded by the curve $y^2 = 4x$, y − axis and the line y = 3 is:

(A) 2 **(B) 9/4**

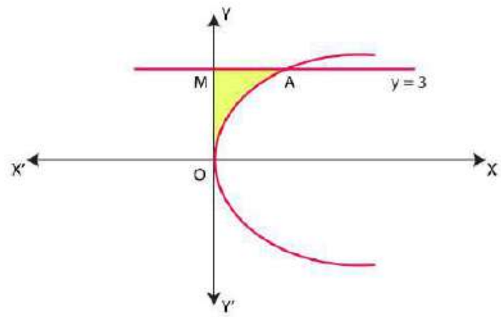
(C) 9/3 **(D) 9/2**

Solution:

Option (B) is correct.

Explanation:

Equation of the curve (parabola) $y^2 = 4x$



Required area $=$ Area OAM $= \left|\int_0^3 x\,dx\right| = \left|\int_0^2 \frac{y^2}{4}\,dy\right|$

$= \frac{1}{4}\left|\left(\frac{y^3}{3}\right)_0^2\right| = \frac{1}{4}\left|\frac{27}{3} - 0\right| = \frac{9}{4}$ sq. units

Exercise 8.2

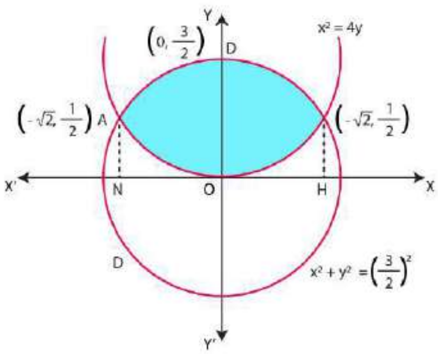
Question 1

Find the area of the area $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$

Solution:

Step 1: Equation of the circle is $4x^2 + 4y^2 = 9$

$x^2 + y^2 = \frac{9}{4}$ ………… (1)



Here, centre of circle is (0, 0) and radius is 3/2

Equation of parabola is $x^2 = 4y$ … … … . (2)

Step 2: To find values of x and y

Put $x^2 = 4y$ in equation (1), $4y + y^2 = \frac{9}{4}$

$16y + 4y^2 = 9$

$4y^2 = 16y - 9 = 0$

$4y^2 = 18y - 2y - 9 = 0$
$2y\,(2y + 9) - (2y + 9) = 0$
$(2y + 9)(2y - 1) = 0$
$2y + 9 = 0$ Or $2y - 1 = 0$
$\Rightarrow y = \frac{-9}{2}$ *or* $y = \frac{1}{2}$

Find the value of x:

Put $y = \frac{-9}{2}$ in $x^2 = 4y,$
$\Rightarrow x^2 = 4y\left(\frac{-9}{2}\right)$ = - 18
Put $y = \frac{1}{2}$ in $x^2 = 4y,$
$\Rightarrow x^2 = 4y\ \left(\frac{1}{2}\right) = 2$
=> $x + \pm 2$
Therefore Points of intersection of circle (1) and parabola (2) are
A $\left(-\sqrt{2}, \frac{1}{2}\right)$ and B $\left(\sqrt{2}, \frac{1}{2}\right)$

Step 3: Area OBM = Area between parabola (2) and y-axis

$$= \int_0^{\frac{1}{2}} x\, dy$$

$\left[\because \text{At O}, y = 0 \text{ and at B}, y\ \frac{1}{2}\right]$

$$= \int_0^{\frac{1}{2}} 2y^{\frac{1}{2}}\, dy$$

$\left[\because x^2 = 4y\ \Rightarrow x = 2\sqrt{y} = 2y^{\frac{1}{2}}\right]$

$=2.\frac{\left(y^{\frac{3}{2}}\right)_0^{\frac{1}{2}}}{\frac{3}{2}} = 2.\frac{2}{3}\left[\left(\frac{1}{2}\right)^{\frac{1}{2}} - 0\right]$

$= \frac{4}{3}.\frac{1}{2}\sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{3}$ ………….. (3)

Step 4: Now area BDM = Area between circle (1) and y –axis

$$= \int_{\frac{1}{2}}^{\frac{1}{2}} x\, dy$$

$\left[\because \text{At B}, y = \frac{1}{2} \text{ and at D}, y = \frac{3}{2}\right]$

$$= \int_{\frac{1}{2}}^{\frac{1}{2}} \left(\frac{3}{2}\right)^2 - y^2\, dy$$

$$\left[\because\ x^2 = \left(\frac{3}{2}\right)^2 - y^2\ \Rightarrow x = \sqrt{\left(\frac{3}{2}\right)^2 - y^2}\right]$$

$$= \left[\frac{y}{2}\sqrt{\left(\frac{3}{2}\right)^2 - y^2} + \frac{\left(\frac{3}{2}\right)^2}{2}\ sin^{-1}\frac{y}{\frac{3}{2}}\right]_{\frac{1}{2}}^{\frac{3}{2}}$$

$$= \frac{3}{4}\sqrt{\left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2} + \frac{9}{8}\ sin^{-1}\frac{\frac{3}{2}}{\frac{3}{2}} - \left[\frac{1}{4}\sqrt{\frac{9}{4} - \frac{1}{4}} + \frac{9}{8}\ sin^{-1}\frac{\frac{1}{2}}{\frac{3}{2}}\right]$$

$$= \left(\frac{3}{4} \times 0\right) + \frac{9}{8}\ sin^{-1}1 - \left[\frac{1}{4}\sqrt{\frac{8}{4}} + \frac{9}{8}\ sin^{-1}\frac{1}{3}\right]$$

$$= \frac{9}{8} \times \frac{\pi}{2} - \frac{1}{4}\sqrt{2} - \frac{9}{8}\ sin^{-1}\frac{1}{3}$$

$$= \frac{9\pi}{16} - \frac{\sqrt{2}}{4} - \frac{9}{8}\ sin^{-1}\frac{1}{3} \ldots\ldots\ldots\ldots (4)$$

Step 5:

Required shaded area = Area AOBDA = 2 (Area OBD) = (Area OBM + Area MBD)

$$= 2\left[\frac{\sqrt{2}}{3} + \left(\frac{9\pi}{16} - \frac{\sqrt{2}}{4} - \frac{9}{8}\ sin^{-1}\frac{1}{3}\right)\right] = 2\left[\sqrt{2}\left(\frac{1}{3} - \frac{1}{4}\right) + \frac{9\pi}{16} - \frac{9}{8}\ sin^{-1}\frac{1}{3}\right]$$

$$= 2\sqrt{2}\left(\frac{4-1}{12}\right) + \frac{9\pi}{8} - \frac{9}{4}\ sin^{-1}\frac{1}{3}$$

$$= \frac{\sqrt{2}}{6} + \frac{9\pi}{8} - \frac{9}{4}\ sin^{-1}\frac{1}{3} = \frac{\sqrt{2}}{6} + \frac{9}{4}\left(\frac{\pi}{2} - sin^{-1}\frac{1}{3}\right)$$

$$= \frac{\sqrt{2}}{6} + \frac{9}{4}cos^{-1}\frac{1}{3}\left[\because \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta\right]$$

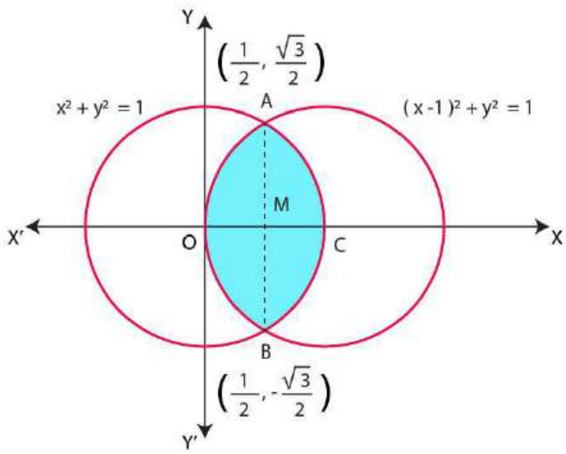
Question 2

Find the area bounded by the curves $(x-1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$

Solution:

Equations of two circles are

$x^2 + y^2 = 1 \ldots\ldots\ldots\ldots\ldots (1)$



And $(x-1)^2 + y^2 = 1$ ………. (2)
From equation (1), $y^2 = 1 - x^2$
Put this value in equation (2),
$(x-1)^2 + 1 - x^2 = 1$
$\Rightarrow x^2 + 1 - 2x + 1 - x^2 = 1$
$\Rightarrow -2x + 1 = 0$
$\Rightarrow x = \frac{1}{2}$
Put $x = \frac{1}{2}$ in $y^2 = 1 - x^2$
$y^2 = 1\left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4} \Rightarrow y = \pm \frac{\sqrt{3}}{4}$
The two points of intersection of circles (1) and (2) are $\left(\frac{1}{2} . \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{2} . \frac{-\sqrt{3}}{2}\right)$
Now from equation (1) y = $\sqrt{1 - x^2}$ in first quadrant and from equation (2) y = $\sqrt{1 - (x-1)^2}$ in first quadrant
Required area OACBO = 2 × Area OAC = 2 (Area OAD + Area DAC)

$$= 2\left[\int_0^{\frac{1}{2}} y \text{ of circle (ii)} dx + \int_{\frac{1}{2}}^{1} y \text{ of circle (i)} dx\right]$$

$$= 2\left[\int_0^{\frac{1}{2}} \sqrt{1 - (x-1)^2}\, dx + \int_{\frac{1}{2}}^{1} \sqrt{1 - x^2}\, dx\right]$$

$$= 2\left[\left\{\frac{(x-1)\sqrt{1 - (x-1)^2}}{2} + \frac{1}{2} \sin^{-1}(x-1)\right\}_0^{\frac{1}{2}} + \left\{\frac{x\sqrt{1 - x^2}}{2} + \frac{1}{2} \sin^{-1} x\right\}_{\frac{1}{2}}^{1}\right]$$

$$= \left\{-\frac{1}{2}\sqrt{\frac{3}{4}} + \sin^{-1}\left(-\frac{1}{2}\right)\right\} - \sin^{-1}(-1) - \left\{\frac{1}{2}\sqrt{\frac{3}{4}} + \sin^{-1}\frac{1}{2}\right\}$$

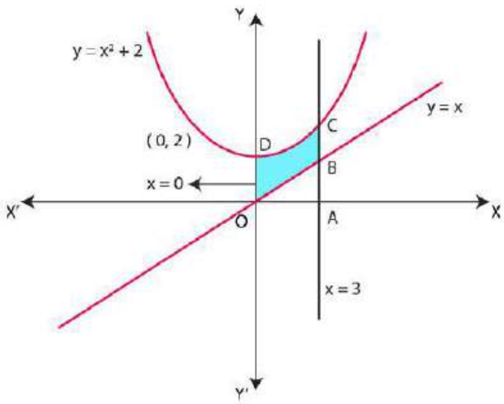
$= -\frac{\sqrt{3}}{4} - \frac{\pi}{6} + \frac{\pi}{2} + \frac{\pi}{2} - \frac{\sqrt{3}}{4} - \frac{\pi}{6} = \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$sq. units

Question 3

Find the area of the region by the curve $y = x^2 + 2.\ y = x, x = 0$ and x = 3.

Solution:

Equation of the given curve is



(Point D is (0,2)
$y = x^2 + 2 \ldots\ldots\ldots\ldots\ldots (1)$
$x^2 = y - 2$
Here vertex of the parabola is (0, 2)
Equation of the given line is y = x …………… (2)

x	0	1	2
y	0	1	2

We know that, slope of straight line passing through the origin is always 1, that means, making an angle of 45 degrees with x- axis.
Here also, Limits of integration area given to be x=0 to x=3.
Area bounded by parabola (1) namely $y = x^2 + 2$. the x-axis and the ordinates x=0 to x=3 is the area OACD and $\int_0^3 y\,dx = \int_0^3 (x^2 + 2)\,dx$
$= \left(\frac{x^3}{3} + 2x\right)_0^3$
= (9 + 6) – 0 = 15 ……….(3)
Again Area bounded by parabola (2) namely y=x the x-axis and the ordinates x=0 to x=3 is the area OAB and

$$\int_0^3 y\,dx = \int_0^3 x\,dx$$

$= \left(\frac{x^2}{2}\right)_0^3 = \frac{9}{2} - 0 = \frac{9}{2}$ ……………….. (3)

Required area = Area OBCD = Area OACD – Area OAB

= Area given by equation (3) – Area given by equation (4)

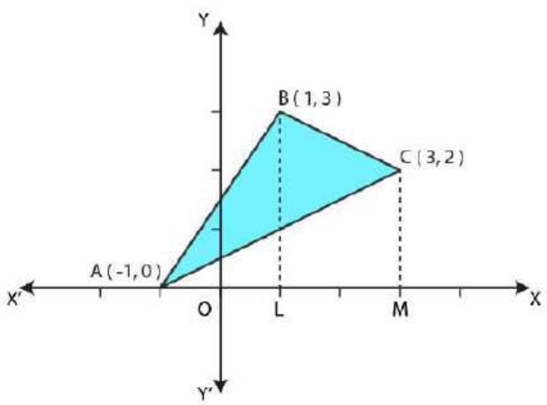
$= 15 - \frac{9}{2} = \frac{21}{2}$ sq. units

Question 4

Using integration, find the area of the region by the triangle whose vertices are (-1, 0), (1,3) and (3,2)

Solution:

Vertices of triangle are A (-1,0), B (1,3) and C (3,2).



Therefore, equation of the line is

$y - 0 \frac{3-0}{1-(-1)} (x - (-1))$

$\left[\because y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x_2 - x_1)\right]$

$y = \frac{3}{2} (x + 1)$

Area of ΔABC = Area bounded by line AB and x – axis

$$= \int_{-1}^{1} y\,dx$$

$[\because \text{At } A, x = -1 \text{ and at } B, x = 1]$

$$= \int_{-1}^{1} \frac{3}{2} (x + 1)\ dx$$

$= \frac{3}{2}\left(\frac{x^2}{2} + x\right)_{-1}^{1}$

$= \frac{3}{2}\left[\left(\frac{1}{2} + 1\right) + \left(\frac{1}{2} - 1\right)\right]$

$= \frac{3}{2}\left(\frac{3}{2} + \frac{1}{2}\right) = \frac{3}{2} . \frac{4}{2} = 3$ ……………. (1)

= Again equation of line BC is $y - 3 = \frac{3}{2} + \frac{1}{2}(x - 1) \Rightarrow y = \frac{1}{2}(7 - x)$

Area of trapezium BLMC = Area bounded by line BC and x – axis

$\Rightarrow \int_1^3 y\,dx = \int_1^3 \frac{1}{2}(7 - x)dx$

$= \frac{1}{2}\left(7x - \frac{x^2}{2}\right)_1^3$

$= \frac{1}{2}\left[\left(21 - \frac{9}{2}\right) - \left(7 - \frac{1}{2}\right)\right]$

$= \frac{1}{2}\left(21 - \frac{9}{2} - 7 + \frac{1}{2}\right) = \frac{1}{2}\left(\frac{42-9-14+1}{2}\right)$

$= \frac{1}{4} \times 20 = 5$ ………… (2)

Again equation of line AC is $y - 0 = \frac{2-0}{3-(-1)}(x - (-1)) \Rightarrow y = \frac{1}{2}(x + 1)$

Area of triangle ACM = Area bounded by line AC and x-axis

$=> \int_{-1}^{3} y\,dx = \int_{-1}^{3} \frac{1}{2}(x + 1)dx$

$= \frac{1}{2}\left[\left(\frac{x^2}{2} + x\right)_{-1}^{3}\right]$

$= \frac{1}{2}\left(\frac{9}{2} + 3 - \frac{1}{2} + 1\right)$

$= \frac{1}{2}\left(\frac{9+6-1+2}{2}\right)$

$= \frac{1}{2} \times 16 = 4$ …………. (3)

Therefore

Required area= Area of ΔABC + Area of Trapezium BLMC – Area ΔACM

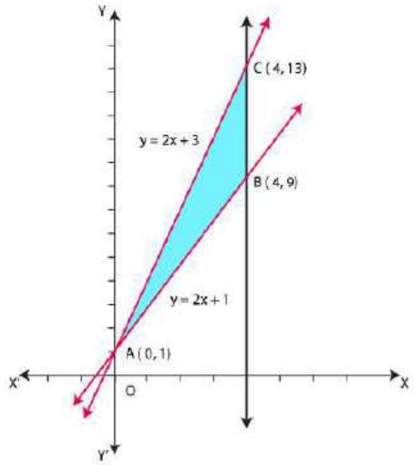
= 3 + 5 – 4 = 4 sq. units

Question 5

Using integration, find the area of the triangular region whose sides have the equations y = 2x + 1, y = 3x + 1 and x=4.

Solution:

Equations of one side of triangle is



$y = 2x\ 1 \ldots\ldots\ldots\ldots (1)$
$y = 3x\ 1 \ldots\ldots\ldots\ldots (2)$ And
$x = 4 \ldots\ldots\ldots.. (3)$
Solving equation (1) and (2), we get x=0 and y=1
So, Point of intersection of lines (1) and (2) is A (0, 1)
Put x=4 in equation (1), we get y=9
So, Point of intersection of lines (1) and (3) is B (4, 9)
Put x=4 in equation (1), we get y=13
Point of intersection of lines (2) and (3) is C (4, 13)
Area between line (2), that is AC and x-axis

$$= \int_0^4 y\,dx = \int_0^4 (3x+1)\,dx = \left(\frac{3x^2}{2} + x\right)_0^4$$

= 24 + 4 = 28 sq. units ……….(iv)
Again Area between line (1) , that is AB and x-axis

$$= \int_0^4 y\,dx = \int_0^4 (2x+1)\,dx$$

= $(x^2 + x)_0^4$
16 + 4 = 20 sq. units ……….(v)
Therefore, required area of ΔABC
= Area given by (4) – Area given by (5)
= 28 – 20 = 8 sq. units

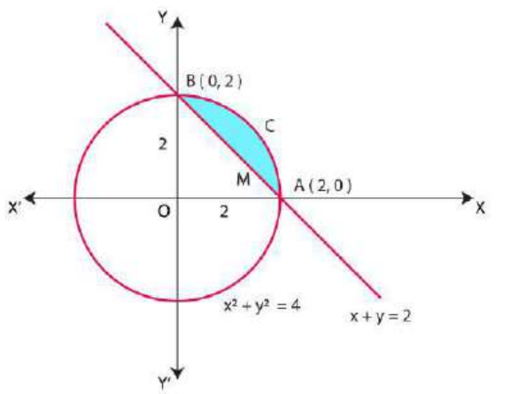
Question 6

Choose the correct answer:
Smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$ is:

(A) $2(\pi - 2)$ **(B) $\pi - 2$**
(C) $2\pi - 1$ **(B) $2\ (\pi + 2)$**

Solution:

Option (B) is correct.
Explanation:
Equation of circle is $x^2 + y^2 = 2^2$ ………….. (1)



$\Rightarrow y = \sqrt{2^2 - x^2}$ …………….. (2)
Also, equation of the line is $x + y = 2$

x	0	2
y	2	0

Therefore graph of equation (3) is the straight line joining the points (0, 2) and (2, 0).
From the graph of circle (1) and straight line (3), it is clear that points of intersections of circle (1) and straight line (3) are A (2, 0) and B (0, 2).
Area OACB, bounded by circle (1) and coordinate axes in first quadrant.

$= \left|\int_0^2 y\,dx\right| = \left|\int_0^2 \sqrt{2^2 - x^2}\,dx\right|$

$= \left(\frac{x}{2}\sqrt{2^2 - x^2} + \frac{2^2}{2}\,sin^{-1}\frac{x}{2}\right)_0^2$

$= \left(\frac{2}{2}\sqrt{4-4} + 2sin^{-1}1\right) - (0 + 2\,sin^{-1}0)$

$= 0 + 2\left(\frac{\pi}{2}\right)$-2 (0) = π sq. units ……….(iv)

Area of triangle OAB bounded by straight line (3) and coordinate axes

$= \left|\int_0^2 y\,dx\right| = \left|\int_0^2 (2-x)dx\right|$

$= \left(2x - \frac{x^2}{2}\right)_0^2$

$= (4-2) - (0,0) = 2$ sq. units …………….. (v)

Required shaded area = Area OACB given by – Area of triangle OAB by (v) $= (\pi - 2)$ sq. units

Question 7

Choose the correct answer:

Area lying between the curves $y^2 = 4x$ and $y = 2x$ is:

(A) $\frac{2}{3}$ **(B) $\frac{1}{3}$**

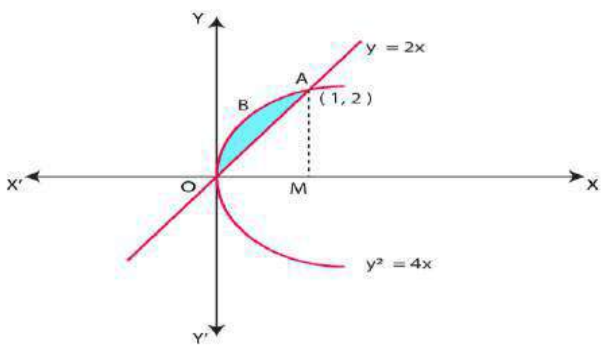
(C) $\frac{1}{4}$ **(C) $\frac{3}{4}$**

Solution:

Option (B) is correct

Explanation:

Equation of curve (parabola) is $y^2 = 4x$ ……….. (1)



$\Rightarrow y = 2\sqrt{x} = 2x\ \frac{1}{2} \ldots \ldots .. (2)$

Equation of another curve (line) is y=2x …..(3)

Solving equation (1) and (3), we get x=0 or x=1 and y=0 or y=2

Therefore, Points of intersections of circle (1) and line (2) are O (0, 0) and A (1, 2).

Now Area OBAM $=$ Area bounded by parabola (1) and x – axis $= \left|\int_0^1 y\,dx\right|$

$= \left|\int_0^1 2x^{\frac{1}{2}}dx\right| = 2\frac{\left(x^{\frac{3}{2}}\right)_0^1}{\frac{3}{2}}$

$= \frac{4}{3}(1-0) = \frac{4}{3} \ldots\ldots..(4)$

Also, Area Δ OAM = Area bounded by parabola (3) and x –axis

$= \left|\int_0^1 y\,dx\right| = \left|\int_0^1 2x\,dx\right| = 2\left(\frac{x^2}{2}\right)_0^1$

$= (1-0) = 1 \ldots\ldots\ldots.(5)$

Now required shaded area OBA = Area OBAM – Area of Δ OAM

$= \frac{4}{3} - 1 = \frac{4-3}{3} = \frac{1}{3}$ sq. units

Miscellaneous Examples

Question 1

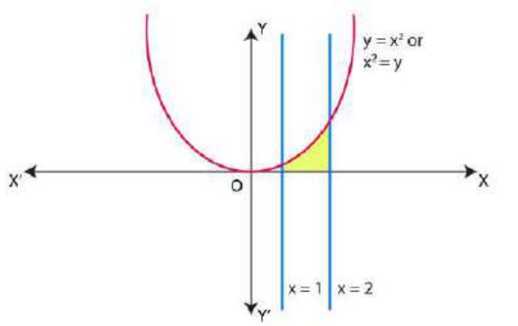
Find the area under of the given curves and given lines:

I. $y = x^2 . x = 1, x = 2$ **and x –axis.**
II. $y = x^4 . x = 1, x = 5$ **and x –axis.**

Solution:

I. **Equation of the curve is**

$y = x^2$ ………… (1)



Require area bounded by curve (1), vertical line x – 1, x =2 and x –axis

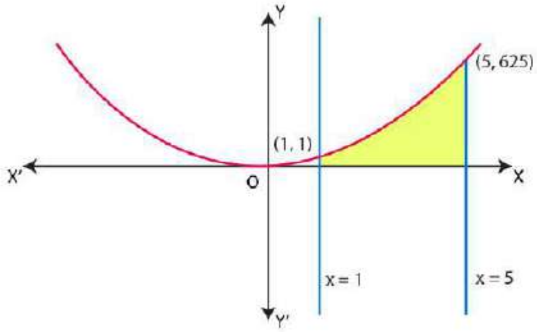
$= \int_1^2 y\,dx$

$= \left(\frac{x^3}{3}\right)_1^2$

$= \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$ sq. unit

II. **Equation of the curve**

$y = x^4$ ………. (1)



It is clear that curve (1) passes through the origin because x = 0 from (1) y = 0.
Table of values for curve $y = x^4$

X	1	2	3	4	5
y	1	16	81	256	625

Required shaded area between the curve$y = x^4$, vertical lines $x = 1, x = 5$ and $x -$ axis

$$= \int_1^5 y\, dx = = \int_1^5 x^4\, dx$$

$$= \left(\frac{x^5}{5}\right)_1^5 = \frac{5^5}{5} - \frac{1^5}{5}$$

$$= \frac{3125-1}{5} = \frac{3124}{5}$$

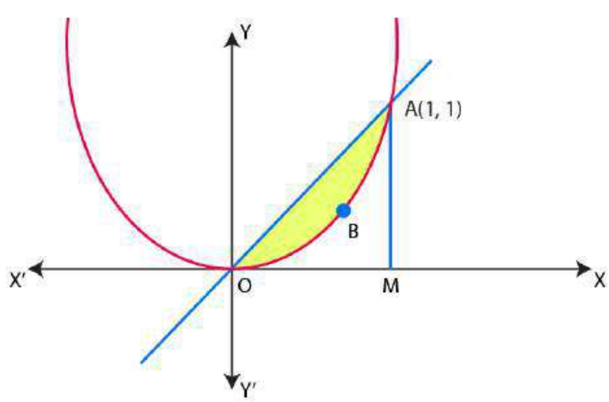
= 624.8 sq units

Question 2

Find the area between the curves the y =x and y =x^2

Solution:

Equation of one curve (straight line) is y = x …………. (i)



Equation of second curve (parabola) is $y = x^2$ .....(ii)
Solving equation (i) and (ii), we get $x = 0$ or $x = 1$ and $y = 0$ or $y = 1$
Points of intersection of line (i) and parabola (ii) are O (0, 0) and A (1, 1).
Now Area of triangle OAM
= Area bounded by line (i) and x-axis

$= \int_0^1 y\,dx = = \int_0^1 x\,dx$

$= \left(\frac{x^2}{2}\right)_0^1$

$= \frac{1}{2} - 0 = \frac{1}{2}$ sq. units

Also Area OBAM = Area bounded by parabola (ii) and x – axis

$= \int_0^1 y\,dx = \int_0^1 x^2\,dx$

$= \left(\frac{x^3}{3}\right)_0^1$

$= \frac{1}{3} - 0 = \frac{1}{3}$ sq. units

Required area OBA between line (i) and parabola (ii)
= Area of triangle OAM – Area of OBM

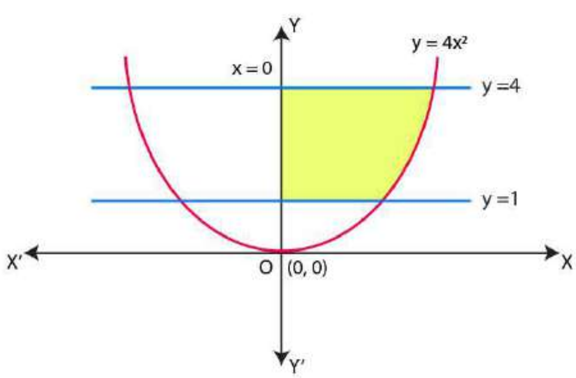
$= \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}$ sq. units

Question 3

Find the area of the region lying in the first quadrant and bounded. by $y = 4x^2$, $x = 0, y = 1$ and $y = 4$

Solution:

Equation of the curve is $y = 4x^2$



$x^2 = \frac{y}{4}$ .... (i)

Or $x = \frac{\sqrt{y}}{2}$ .... (ii)

Here required shaded area of the region lying in first quadrant bounded by parabola (i) $x = 0$ and the horizontal lines$y = 1$ and $y = 4$ is.

$$\int_1^4 x\,dy = \int_1^4 \frac{\sqrt{y}}{2}\,dy = \frac{1}{2}\int_1^4 y^{\frac{1}{2}}$$

$$= \frac{1}{2}\left|\frac{\left(y^{\frac{3}{2}}\right)_1^4}{\frac{3}{2}}\right|$$

$$= \frac{1}{2}\cdot\frac{2}{3}\left(4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right)$$

$$= \frac{1}{3}\left(4\sqrt{4} - 1\right)$$

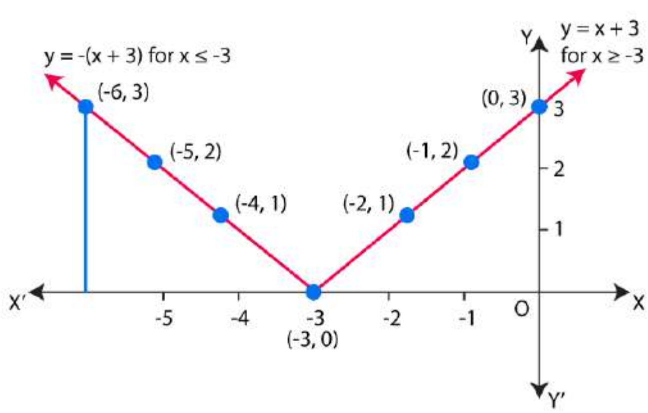
$$= \frac{1}{3}(8 - 1) = \frac{7}{3} \text{ sq. units}$$

Question 4

Sketch the of $y = |x + 3|$ and evaluate $\int_{-6}^{0} |x + 3|\,dy$

Solution:

Equation of the given curve is y= $|x + 3|$............ (i)



y= $|x + 3| \geq 0$ for all real x.

Graph of curve is only above the x –axis I.e., in first and second quadrant only.

y= $|x + 3|$

$= x + 3$

If $x + 3 \geq 0$

$x \geq -3$ .... (ii)

And y= $|x+3|$
$= -(x+3)$
If $x+3 \le 0$
$x \le -3$ ……….. (iii)
Table of values for $y = x+3$ for $x \ge -3$

x	Y
-3	0
-2	1
-1	2
0	3

Table of values for $y = x+3$ for $x \le -3$

x	Y
-3	0
-4	1
-5	2
-6	3

Now $\int_{-6}^{0} |x+3|\, dx$

$= \int_{-6}^{-3} |x+3|\, dx + \int_{-3}^{0} |x+3|\, dx$

$= \int_{-6}^{-3} -(x+3)\, dx + \int_{-3}^{0} (x+3) dx$

$= \left(\frac{x^2}{2} + 3x\right)_{-6}^{-3} + \left(\frac{x^2}{2} + 3x\right)_{-3}^{0}$

$= \left[\frac{9}{2} - 9 - (18-18)\right] + \left[0 - \left(\frac{9}{2} - 9\right)\right]$

$= \frac{9}{2} + 9 + 0 + 0 - \frac{9}{2} + 9$

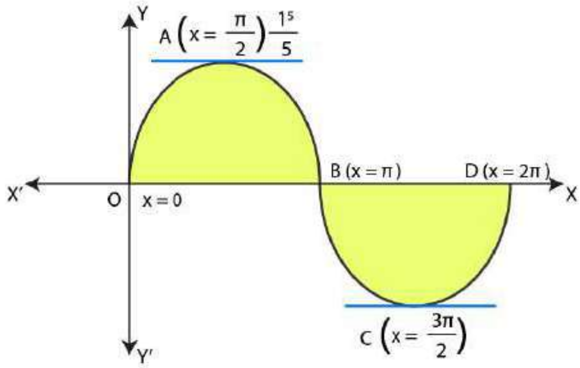
$= 18 - \frac{18}{2} = 18 - 9 = 9$ sq. units

Question 5

Find that area bounded by the curve y = sin x between x=0 and x=2π.

Solution:

Equation of the curve is y = sin x ………………. (i)
$y = \sin x \geq 0$ for $0 \leq x \leq \pi$: as graph is in I and II quadrant
And $y = \sin x \geq 0$ for $\pi \leq x \leq 2\pi$: as graph is in III and IV quadrant



If tangent is parallel to x- axis, then
$\frac{dy}{dx} = 0$
$\Rightarrow \cos x = 0$
$\Rightarrow x = \frac{\pi}{2}.\frac{3\pi}{2}$
Table of values of curve y = sin x between x = 0 and x 2π

X	y
0	0
$\boldsymbol{\frac{\pi}{2}}$	1
$\boldsymbol{\pi}$	0
$\boldsymbol{\frac{3\pi}{2}}$	-1
$\boldsymbol{2\pi}$	0

Now required shaded area = Area OAB + Area BCB

$= \int_0^{\pi} y\, dx + \int_{\pi}^{2\pi} y\, dx$

$= \int_0^{\pi} sin\, dx + \int_{\pi}^{2\pi} \sin x\ dx$

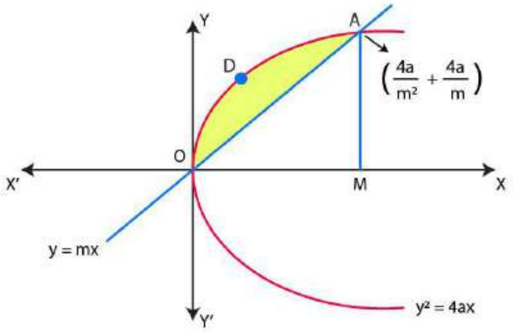
$= -(\cos x)_0^{\pi} + (\cos x)_0^{2\pi}$
$= -1(-1-1), + -(1+1)$
$= 2 + 2 = 4$ sq. units

Question 6

Find the area enclosed by the parabola $y^2 = 4ax$ and the y =mx.

Solution:

Equation of parabola is $y^2 = 4ax$ .... (i)



The area enclosed between the parabola and line is the shaded area OADO.
From figure: And the points of intersection of curve and line
0, (0,0)and A $\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$
Now Area ODAM = Area of parabola and x-axis

$$= \int_0^{\frac{4a}{m^2}} 2\sqrt{a}x^{\frac{1}{2}}\,dx$$

$$= 2\sqrt{a}\,\frac{\left(x^{\frac{3}{2}}\right)_0^{\frac{4a}{m^2}}}{\frac{3}{2}}$$

$$= \frac{4\sqrt{a}}{3}\left(\frac{4a}{m^2}\right)^{\frac{3}{2}}$$

$$= \frac{32\,a^2}{3m^3}$$ .... (ii)

Again ΔOAM = Area between line and x=axis

$$= \left|\int_0^{\frac{4a}{m^2}} mx\,dx\right| = m\left(\frac{x^2}{2}\right)_0^{\frac{4a}{m^2}}$$

$$= \frac{m}{2}\left(\left(\frac{4a}{m^2}\right)^2 - 0\right)$$

$$= \frac{m}{2}\cdot\frac{16a^2}{m^4} = \frac{8a^2}{m^2}$$ .... (ii)

Requires shaded area = Area ODAM – Area of ΔOAM

$$\frac{32a^2}{3m^3} = \frac{8a^2}{m^3}$$

$= \frac{a^2}{m^2}\left(\frac{32}{3} - 8\right)$

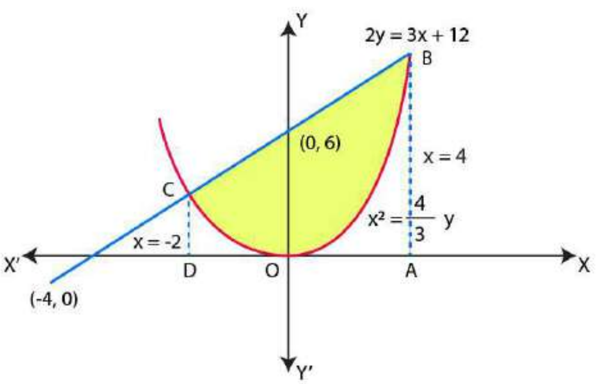
$= \frac{8a^2}{3m^3}$

Question 7

Find the area enclosed by the parabola $4y = 3x^2$ and the line $2\ y = 3\ x + 12$.

Solution:

Equation of the parabola is
$4y = 3x^2$ .... (i)
Or $x^2 = \frac{4}{3}\ y$



Equation of the line is $2\ y = 3\ x + 12$ .... (ii)
From graph, points of intersection are B (4, 12) and C (-2, 3).

Now, Area ABCD $= \left| \int_{-2}^{4} \left(\frac{3}{2}\ x + 6\right)\ dx \right|$

$= \left[\frac{3}{4}\ x^2 + \ 6x\right]_{-2}^{4}$

$= (12 + 240 - (3 - 12)$

= 45 sq. units

Again, Area COD $+$ Area OAB $= \int_{-2}^{4} \left(\frac{3}{4}\ x^2\right) dx$

$= \frac{1}{4}\ [64 - \ (-8)]$ = 18 sq. units

Therefore,

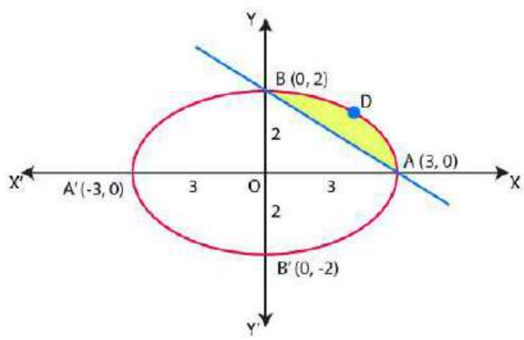
Requirement area = Area ABCD – (Area COD+ Area OAB)
$= 45 - 18 = 27$ sq. units

Question 8

Find the area of the smaller region bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the line $\frac{x}{3} + \frac{y}{2} = 1$

Solution:

Equation of the ellipse is
$\frac{x^2}{9} + \frac{y^2}{4} = 1$ ……………… (i)



Here points of intersection of ellipse (i) with a-axis are
A (3, 0) and A' (-3, 0) and intersection of ellipse (i) with y –axis B (0, 2) and B, (0, 2).
Also, the points of intersection of ellipse (i) and line $\frac{x}{3} + \frac{y}{2} = 1$area A (3, 0) and B (0, 2).
Therefore

$$= \int_0^3 \frac{2}{3}\sqrt{9 - x^2}\, dx$$

$$= \frac{2}{3}\left[\frac{x}{2}\sqrt{3^2 - x^2}\, \frac{3^2}{2}\, sin^{-1}\frac{x}{3}\right]$$

$$= \frac{2}{3}\left[\frac{3}{2}\sqrt{9 - \;\, 9} + \frac{9}{2}\, sin^{-1} 1 - \left(0 + \frac{9}{2} sin^{-1} 0\right)\right]$$

$= \frac{2}{3}.\frac{9\pi}{4} = \frac{3\pi}{2}$ sq. units ………. (ii)

Again Area of triangle OAB = Area bounded by line AB x-axis

$$= \int_0^3 \frac{2}{3}\sqrt{3 - x}\, dx$$

$= \frac{2}{3}\left\{\left(9 - \frac{9}{2}\right) - 0\right\}$

$= \frac{2}{3}.\frac{9}{2} = \;3$ sq. units ………. (ii)

Now required shaded area = Area OADM – Area OAB

$= \frac{3\pi}{2} - 3$

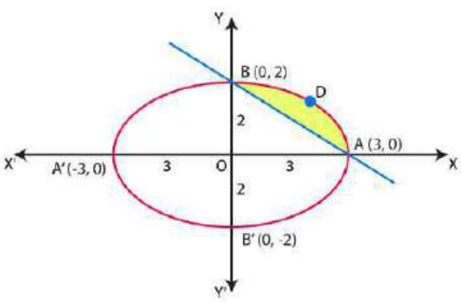
$= 3\left(\frac{\pi}{2} - 1\right) = \frac{3}{2}(\pi - 2)$ sq. units

Question 9

Find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$and the line $\frac{x}{a} + \frac{y}{b} = 1$.

Solution:

Equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .... (i)



Area between arc AB of the ellipse and x axis

$$= \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx$$

$$= \frac{b}{a}\left[\frac{x}{a}\sqrt{a^2 - x^2} + \frac{a^2}{a}\, sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{b}{a}\left[0 + \frac{a^2}{2}\, sin^{-1} 1\, (0 + 0)\right]$$

$= \frac{b}{a} \cdot \frac{a^2}{2} \cdot \frac{\pi}{2} = \frac{\pi\, ab}{4}$ .... (ii)

Also Area between chord and x-axis

$$= \int_0^a \frac{b}{a}(a - x)\, dx$$

$$= \frac{b}{a}\left[ax - \frac{x^2}{2}\right]_0^a$$

$$= \frac{b}{a}\left(a^2 - \frac{a^2}{2}\right)$$

$$= \frac{b}{a} \cdot \frac{a^2}{2} = \frac{1}{2}ab$$

Now, Required area = (Area between arc AB of the ellipse and x-axis) – (Area between chord AB and x-axis)

$= \frac{\pi ab}{4} = \frac{ab}{2} = \frac{ab}{4}\ (\pi - 2)$sq. units

Question 10

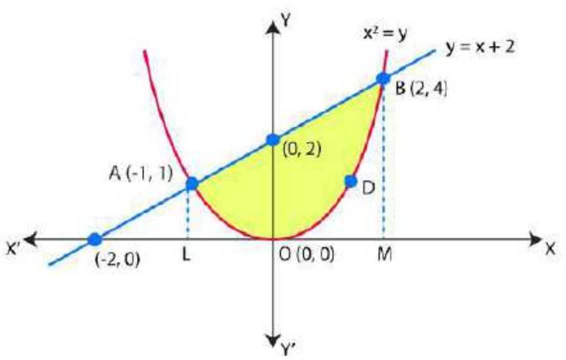
Find the area of the region enclosed by the parabola x2 = y, the line y = x+2 and x- axis.

Solution:

Equation of parabola is $x^2 = y$ ………… (i)
Equation of line is y = x+2 ………… (ii)
Here the two points of intersections of parabola (i) and line (ii) are A (-1, 1) and B (2, 4).
Area ALODBM = Area bounded by parabola (i) and x-axis

$$= \int_{-1}^{2} x^2\ dx = \left(\frac{x^3}{3}\right)_{-1}^{2}$$

$= \frac{8}{3} + \frac{1}{3} = \frac{9}{3} = 3$ sq. units



Also, Area of trapezium ALMB = Area bounded by line (ii) and x-axis

$$= \int_{-1}^{2} (x - 2)\ dx = \left(\frac{x^2}{2}\ 2\ x\right)_{-1}^{2}$$

$= 2 + 4 - \left(\frac{1}{2} - 2\right)$

$= \frac{15}{2}$ sq. units

Now required area = Area of trapezium ALMB – Area ALODBM

$= \frac{15}{2} - 3 = \frac{9}{2}$ sq. units

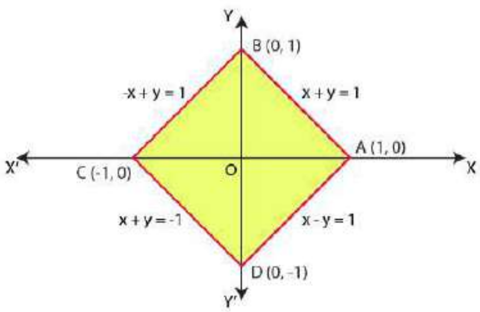
Question 11

Using the method of integration, find the area enclosed by the curve $|x| + |y| = 1$.

[Hint: The required is bounded by lines $x + y = 1, x - y = 1, -x + y = 1$ and $-x - y = 1$].

Solution:

$|x| + |y| = 1$ ………… (i)



The area bounded by the curve (i) is represented by the shaded region ABCD.
The curve intersects the axes at points A (1, 0), B (0, 1), C(-1, 0) and D(0, -1)
As, given curve is symmetrical about x-axis and y-axis.
Area bounded by the curve = Area of square ABCD = 4 x ΔOAB

$$= 4\int_0^1 (1 - x)\, dx$$

$$= 4\left(x - \frac{x^2}{2}\right)_0^1$$

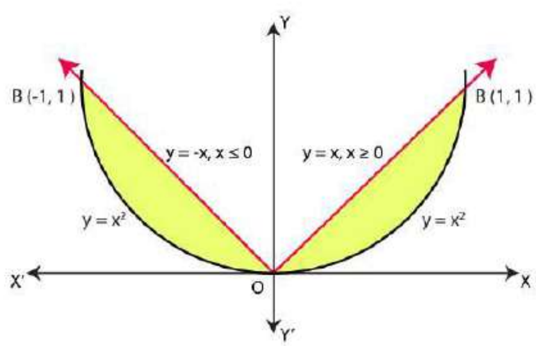
$$= 4 \times \frac{1}{2} = 2 \text{ sq units}$$

Question 12

Find the area bounded by the curves $\{(x, y): y \geq x^2 \text{ and } y = |x|\}$

Solution:

The area bounded by the curves $\{(x, y): y \geq x^2 \text{ and } y = |x|\}$ is represented by the shaded region.



Since area is symmetrical about y –axis

Therefore, required area = Area between parabola and x –axis between limits x = 0 and x = 1

$$= \int_0^1 y\,dx = \int_0^1 x^2\,dx$$

$$= \left(\frac{x^3}{3}\right)_0^1 = \frac{1}{3} \text{ ………… (i)}$$

And Area of ray – y=x and x-axis,

$$= \int_0^1 y\,dx = \int_0^1 x\,dx = \left(\frac{x^2}{2}\right)_0^1 = \frac{1}{2} \text{ … … … … …. (ii)}$$

Required shaded area in first quadrant

= (Area between ray y = x for $x \geq 0$ and x –axis) – (Area between parabola $y = x^2$ and x-axis in first quadrant)

= Area given by equation (ii) – Area given by equation (i)

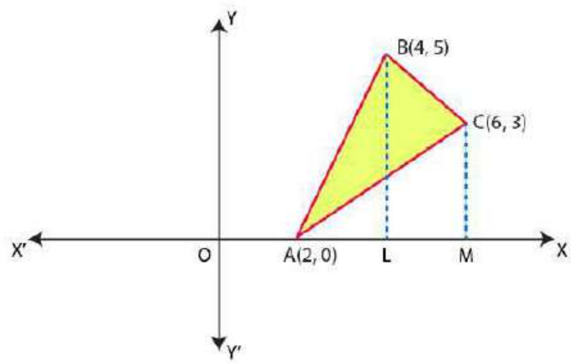
$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ sq. units

Question 13

Using the method of integration, find the area of the triangle whose vertices are A (2, 0), B (4,5) and (6,3)

Solution:

Vertices of the given triangle are A (2, 0), B (4,5) and C (6,3)



Equation of side AB is $y - 0 = \frac{5-0}{4-2}(x-2)$

$$= y = \frac{5}{2}(x-2)$$

Equation of side BC is $y - 5 = \frac{3-5}{6-4}(x-4)$

$= y = 9 - x$

Equation of side AC is $y - 0 = \frac{3-0}{6-2}(x-2)$

$= y = \frac{3}{4}(x-2)$

Now, required shaded, area = Area ΔALB + Area of trapezium BLMC – Area ΔAMC

$$= \int_2^4 \frac{5}{2}(x-2)\,dx + \int_4^6 (9-2)\,dx - \int_2^6 \frac{3}{4}(x-2)\,dx$$

$$= \left[\frac{5}{2}(8-8) - (2-4)\right] + |54-18-(36-8)| - \left[\frac{3}{4}\{18-12-(2-4)\}\right]$$

$$= \frac{5}{2}(0+2) + |36-36+8| - \frac{3}{4}(6+2)$$

$= 5+8-6 = 7$ sq. units

Question 14

Using the method of integration find the find the area of the region bounded lines $2x+y=4$, $3x-2y=6$ and $x-3y+5=0$.

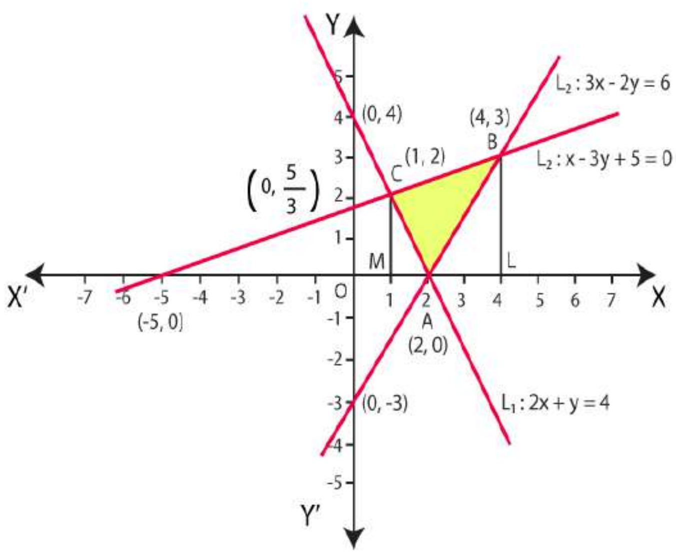
Solution:

Let say, equation of one line l_1 is $2x+y=4$

Equation of second line l_2 is $3x-2y=6$

And Equation of third line l_3 is $x-3y+5=0$

Draw all the lines on the coordinate plane we get



Here, vertices of triangle ABC are A (2,0), B (4,3) and C (1,2).

Now, Required area of triangle = Area of trapezium CLMB – Area ΔABL

$$= \int_1^4 \frac{1}{3}(x+5)\,dx - \int_1^2 (4-2)\,dx - \int_2^4 \frac{3}{2}(x-2)\,dx$$

$= \frac{1}{3}\left[8 + 20 - \left(\frac{1}{2} + 5\right)\right] - \{(8 - 4) - (4 - 1)\} - \frac{3}{2}\,|(8 - 8) - (2 - 4)|$

$= \frac{1}{3}\left(28 - \frac{11}{2}\right) - (4 - 3) - \frac{3}{2} \times 2$

$= \frac{1}{3} \times \frac{45}{2} - 1 - 3$

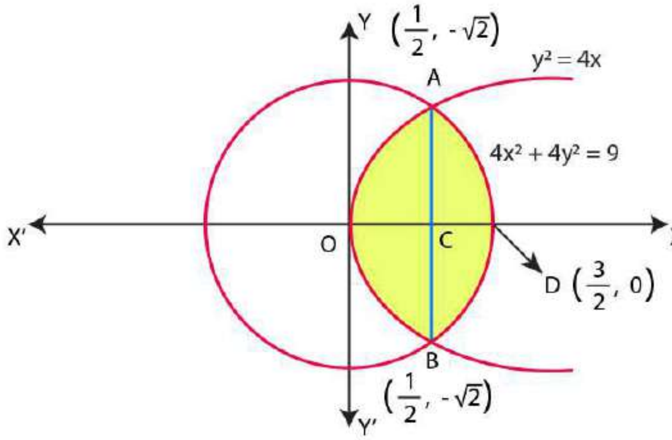
$= \frac{15}{2} - 1 - 3 = \frac{7}{2}$ sq. units

Question 15

Find the area of the region $\{(x, y): y^2 \le 4x \text{ and } 4x^2 + 4y^2 \le 9\}$

Solution:

Equation of parabola is $y^2 = 4x$ … … … … (i)
And equation of circle is $4x^2 + 4y^2 \le 9$……….. (ii)



From figures, points of intersection of parabola (i) and circle (ii) are
A$\left(\frac{1}{2}, \sqrt{2}\right)$ and B $\left(\frac{1}{2}, \sqrt{2}\right)$
Required shaded area OADBO (Area of the circle which is interior to the parabola)
= 2 × Area OADO = 2 [Area OAC + Area CAD]

$= 2\left[\int_0^{\frac{1}{2}} 2\sqrt{x}\,dx \int_{\frac{1}{2}}^{\frac{3}{2}} \sqrt{\frac{9}{4} - x^2}\,dx\right]$

$$= \left[\left\{ 2.\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right\}_0^{\frac{1}{2}} + \left\{ \frac{x\sqrt{\frac{9}{4} - x^2}}{4} + \frac{\frac{9}{4}}{2} sin^{-1} \frac{x}{3/2} \right\}_{\frac{1}{2}}^{\frac{3}{2}} \right]$$

$$= 2\left[\frac{4}{3} \times \frac{1}{2\sqrt{2}} + \frac{9}{8} sin^{-1} 1 - \frac{\frac{1}{2}\sqrt{2}}{2} \frac{9}{8} sin^{-1} \frac{1}{3}\right]$$

$$= 2\left[\frac{\sqrt{2}}{3} + \frac{9}{8}.\frac{\pi}{2} - \frac{\sqrt{2}}{4} - \frac{9}{8} sin^{-1} \frac{1}{3}\right]$$

$$= \left(\frac{9\pi}{8} - \frac{9}{4} sin^{-1} \frac{1}{3} + \frac{\sqrt{2}}{6}\right) \text{ sq units.}$$

Question 16

Choose the correct answer:

Area bounded by the curve $y = x^2$ the $x -$ axis and the ordinate $x = 2$ and $x = 1$ is:

(A) -9 **(B) -15/4**

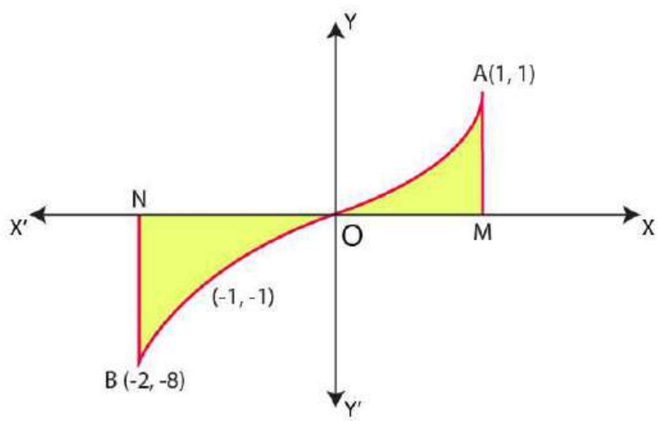
(C) 15/4 **(D) 17/4**

Solution:

Option (D) is correct

Explanation:

Equation of the curve is $y = x^3$



Now, Area OBN ($y = x^3$ for $-2 \le x \le 0$)and Area OAM ($y = x^3$ for $0 \le x \le 1$)

Therefore, required area = Area OBN + Area OBN + Area OAM

$$= \int_{-2}^{0} x^3\, dx + \int_{0}^{1} x^3\, dx$$

$$= \frac{17}{4} \text{ sq units}$$

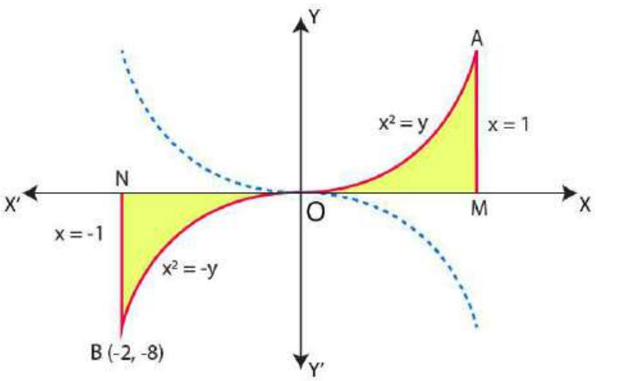
Question 17

Choose the correct answer:
The area bounded by the curve y = x|x|,x- axis and the ordinates x = -1 and x = 1 is given by

(A) 0 **(B) 1/3**
(C) 2/3 **(D) 4/3**

Solution:

Equation of the curve is



$y = x|x| = x\,(x) = \; x^2$ if $x \;\geq\; 0$ .... (1)
And $y = x|x| = x\,(-x) = \; -x^2$ if $x \;\geq\; 0$ .... (2)
Required area = Area OAMO

$$= \int_{-1}^{0} -x^2 \, dx + \int_{0}^{1} x^2 \, dx$$

$= 2/3$ sq. units

Question 18

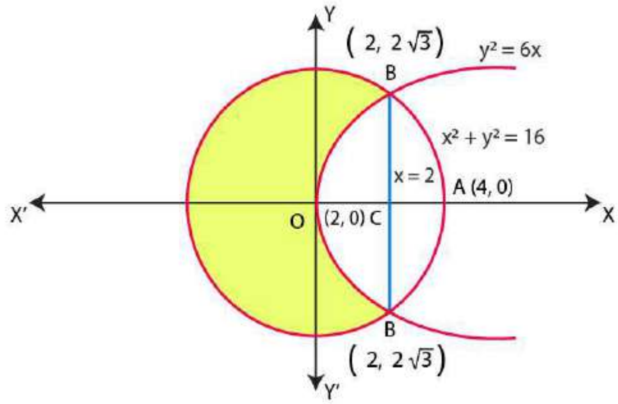
Choose the correct answer:
The area of the circle $x^2 + \; y^2 = 16$ exterior to the parabola $y^2 = 6x$.

(A) $\frac{4}{3}\left(4\,\pi - \sqrt{3}\right)$ **(B)** $\frac{4}{3}\left(4\,\pi + \sqrt{3}\right)$
(C) $\frac{4}{3}\left(8\,\pi - \sqrt{3}\right)$ **(D)** $\frac{4}{3}\left(8\,\pi + \sqrt{3}\right)$

Solution:

Option (c) is correct
Explanation:
Equation of the circle is $x^2 + \; y^2 = 16$ .....(1)
Thus, radius of circle is 4

This circle is symmetrical about x-axis and y- axis.
Here two points of intersection are B $(2, 2\sqrt{3})$ and B' $(2, -2\sqrt{3})$.



Required area = Area of circle – Area of circle interior to the parabola
= πr^2 - Area OBAB'O
= 16π - 2 x Area OBACO
= 16π - 2[Area OBCO + Area BACB]

$$= 16\pi - 2\left[\int_0^2 \sqrt{6x}\ dx + \int_2^4 \sqrt{16 - x^2}\ dx\right]$$

$$= 16\pi - 2\left[\frac{2}{3}\sqrt{6}\,(2\sqrt{2}) + 8\sin^{-1} 1\sqrt{12} - 8\sin^{-1}\frac{1}{2}\right]$$

$$= 16\pi - 2\left[\frac{8}{\sqrt{3}} + 8.\frac{\pi}{2} - 2\sqrt{3} - 8.\frac{\pi}{6}\right]$$

$$= 16\pi - 2\left[\frac{8}{\sqrt{3}} - 2\sqrt{3} + 8\pi\left(\frac{1}{2} - \frac{1}{6}\right)\right]$$

$$= 16\pi - 2\left[\frac{2}{\sqrt{3}} + \frac{8\pi}{3}\right]$$

$$= 16\pi\left(1 - \frac{1}{3}\right) - \frac{4}{\sqrt{3}}$$

$$= \frac{4}{3}\left(8\pi - \sqrt{3}\right) \text{ sq. units}$$

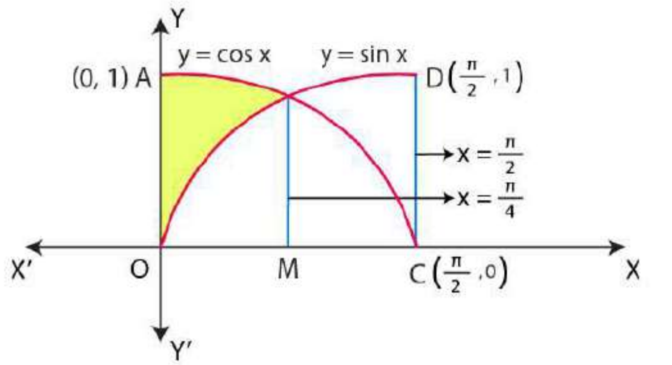
Question 19

Choose the correct answer:
The area bounded by the y –axis, y = cos x and y = sin x when $0 \le x \le \frac{\pi}{2}$ is:

(A) $2\left(\sqrt{2} - 1\right)$ **(B) $\sqrt{2} - 1$**
(C) $\sqrt{2} + 1$ **(D) $\sqrt{2}$**

Solution:

Option (B) is correct
Explanation:
Graph of both the function are intersect at the point
B $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$



Required shaded Area = Area OABC – Area OBC
= Area OABC – (Area OBM + Area BCM)

$$= \int_0^{\pi/2} \cos x \, dx - \left(\int_0^{\pi/4} \sin x \, dx + \int_{\pi/4}^{\pi/2} \cos x \, dx \right)$$

$$= \left(\sin\frac{\pi}{2} - \sin 0^0\right) - \left(-\cos\frac{\pi}{4} + \cos 0^0 + \sin\frac{\pi}{2} - \sin\frac{\pi}{4}\right)$$

$$= 1 + \frac{1}{\sqrt{2}} - 1 - 1 + \frac{1}{\sqrt{2}} = \left(\sqrt{2} - 1\right) \text{ sq. units}$$