Chapter 5
Continuity and Differentiability

Question 1

Prove that the function f(x) = 5x - 3 is continuous at x = 0 at x = -3 and at x = 5.

Solution:

Given function is f(x) = 5x – 3
Continuity at x = 0,
$\lim_{x \to 0} f(x) \lim_{x \to 0} (5x - 3)$
= 5 (0) – 3
= 0 – 3
= – 3
Again, f (0) = 5 (0) – 3 = 0 – 3 = – 3
$\lim_{x\to 0} f(x) = f(x)$, therefore, f (x) is continuous at x = 0
Continuity at x = -3,
$\lim_{x \to -3} f(x) \lim_{x \to -3} (5x - 3) = 5(-3) - 3 = -18$
And $f(-3) = 5(-3) - = -18'$
As $\lim_{x \to -3} f(x) = f(x)$, therefore, f (x)is continuous at x $= -3$
Continuity at x = 5
$\lim_{x \to 5} f(x) \lim_{x \to 5} (5x - 3)$
= 5 (5) -3 = 22
And f(5) = 5 (5) -3 =22
Therefore $\lim_{x \to 5} f(x) = f(x)$, So f(x) is continuous at x = -5

Question 2

Examine the continuity of the function f(x) = 2x^2 - 1 at x = 3.

Solution:

Give function f(x) =2 $x^2 - 1$
Check continuity at x =3,
$\lim_{x \to 3} f(x) \lim_{x \to 3} (2x^2 - 1)$
= 2(3) ^2 – 1 = 17
Therefore $\lim_{x \to 3} f(x) = f(x)$, So f(x) is continuous at x = 3.

Question 3

Examine the following functions for continuity:

a) $f(x) = x - 5$

b) $f(x) = \frac{1}{x-5}, x \neq 5$

c) $f(x) = \frac{x^2 - 25}{x-5}, x \neq -5$

d) $f(x) = |x - 5|$

Solution:

a) Given function $f(x) = x - 5$

We know f is defined at every real number k and its value at k is k -5.

Also observed that $\lim_{x \to k} f(x) = \lim_{x \to k}(x - 5) = k - y = f(k)$

As $\lim_{x \to k} f(x) = f(k)$, therefore, $f(x)$ is continuous at every real number and it is a continuous function.

b) Give function is $f(x) = \frac{1}{x-5}, x \neq 5$

For any real number k ≠5, we have

$$\lim_{x \to k} f(x) = \lim_{x \to k} \frac{1}{x-5} = \frac{1}{k-5}$$

And $f(k) = \frac{1}{k-5}$

$As, \lim_{x \to k} f(x) = f(k)$

Therefore

f(x) is continuous of every point of domain of f and it is a continuous function.

c) Given function is $f(x) = \frac{x^2 - 25}{x+5}, x \neq -5$

For any real number k ≠5, we get

$$\lim_{x \to k} f(x) = \lim_{x \to k} \frac{x^2 - 25}{x+5} = \lim_{x \to k} \frac{(x+5)(x-5)}{x+5} = \lim_{x \to k}(x - 5) = k - 5$$

and $f(k) = \frac{(k+5)(k-5)}{k+5} = k - 5$

As, $\lim_{x \to k} f(x) = f(k), therefore, f(x)$is continous at every point of domain of f and it is a continuous Function

d) Given function is $f(x) = |x - 5|$

Domain *f (x)* is real and infinite for all real x

Here *f (x)* = $|x - 5|$is a modulus function.

As every modulus function continuous

Therefore, f is continuous on its domain R.

Question 4

Prove that the function $f(x) = x^n$ is continuous at $x = n$ where n is a positive integer.

Solution:

Given function is $f(x) = x^n$ where n is a positive integer.

Continuous at x = n, $\lim_{x \to n} f(x) = \lim_{x \to n} (x^n) = n^n$

And $f(n) = n^n$

As, $\lim_{x \to n} f(x) = f(x)$, therefore $f(x)$ is continuous at x = n

Question 5

Is the function *f* defined by $f(x) \begin{cases} x. if\ x \le 1 \\ 5. if\ x > 1 \end{cases}$ continuous at x = 0 at x = 1 at x = 2?

Solution:

Given function is $f(x) \begin{cases} x. if\ x \le 1 \\ 5. if\ x > 1 \end{cases}$

Step 1: At x = 0, we know that *f* is defined at 0 and its value 0.

Then $\lim_{x \to 0} f(x) = \lim_{x \to} x = 0\ and\ f(0) = 0$

Therefore, $f(x)$ is continuous at $x = 0$

Step 2: At x = 1, left hand limit (LHL)0f f $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x) = 1$

Right hand limit (RHL) of f $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x) = 5$

Here $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^-} f(x)$

Therefore, *f* (x) is not continuous at x = 1

Step 3: At $x = 2$, f is defined at 2 and its value at 2 is 5.

$\lim_{x \to 2} f(x) = \lim_{x \to 2} (5) = 5\ therefore\ \lim_{x \to 2} f(x) = f(2)$

Therefore, *f* (x) is not continuous at x = 2

Find all points of discontinuity of *f* where *f* is defined by:

Question 6

$f(x) = \begin{cases} 2x + 3. x \le 2 \\ 2x - 3. x > 2 \end{cases}$

Solution:

Given function is $f(x) = \begin{cases} 2x + 3. x \le 2 \\ 2x - 3. x > 2 \end{cases}$

Here f (x) is defined for $x \le 2$ or $(-\infty. 2)$ and also for $x > 2$ or $(2. \infty)$

Therefore, Domain of f is $(-\infty. 2) \cup (2. \infty) = (-\infty. \infty) = R$

Therefore for all $x < 2. f(x) = 2x + 3$ is a polynomial and hence continuous and for all $x > 2. f(x) = 2x - 3$ is a continuous and hence it is also continuous on R - {2}.

Now left hand limit = $\lim_{x \to 2-} f(x) = \lim_{x \to 2-} (2x + 3) = 2 \times 2 + 3 = 7$

Now left hand limit = $\lim_{x \to 2+} f(x) = \lim_{x \to 2+} (2x + 3) = 2 \times 2 - 3 = 1$
As $\lim_{x \to 2-} f(x) = \lim_{x \to 2+} f(x)$
Therefore $\lim_{x \to 2} f(x)$ does not exist and hence f (x) is discontinuous at only x = 2

Find all points of discontinuity of *f*. Where *f*. is defined by:

Question 7

$$f(x) = \begin{cases} |x| + 3, & \text{if } x \le -3 \\ -2x. & \text{if } -3 < x < 3 \\ 6x + 2, & \text{if } x \ge 3 \end{cases}$$

Solution:

givn function is f (x) = $\begin{cases} |x| + 3, & \text{if } x \le 2 \\ -2x. & \text{if } x > 2 \\ 6x + 2, & \text{if } x \ge 3 \end{cases}$
Here $f(x)$ is defined for $x \le -3$ or $(-\infty.-3)$and for $-3 < x < 3$ and also for $x \ge 3$ or $(3.\infty)$.
Therefore, Domain of f is $(-\infty.-3) \cup (-3.3) \cup (3.\infty) = (-\infty.\infty) = R$
Therefore, for all $x < -3. f(x) = |x| + 3 = -x + 3$ is a polynomial and hence and for all x $(-3 < x < 3. f\ x = -2x$ is a continuous and continuous function and also for all $x > 3. f\ x = 6x + 2$
Therefore f(x) is continuous on R -{−3,3}.
And, x = -3 an x = 3 are partitioning of domain R.
Now, left hand limit = $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (|x| + 3) \lim_{x \to 3^-} (-x + 3) = 3 + 3 = 6$

Right hand limit = $f \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (-2x) = (-2)(-3) = 6$
And $f(-3) = |-3| + 3 = 3 + 3 = 6$
Therefore, $f(x)$ is continuous at $x = 3$
Again left hand limit $\lim_{k \to 3^+}$ f (x) $\lim_{k \to 3^-} (-2x)) = -2(3) = -6$
Right hand limit = $\lim_{k \to 3^+} f$ (x) $\lim_{k \to 3^-} (6x + 2)) = 6(3) + 2 = 20$
As $\lim_{x \to 3^-} f$ (x) $\neq \lim_{x \to 3^+} f$ (x)
Therefore $\lim_{x \to 3} f$ (x) does not exist and hence $f(x)$ is discontinuous at only x = 3

Find all points of discontinuous of *f* where *f* is defined by:

Question 8

$$f(x) = \begin{cases} \frac{|x|}{x}, & if\ x \neq 0 \\ 0, & if\ x = 0 \end{cases}$$

Solution:

Given function is $f\ (x) = \begin{cases} \frac{|x|}{x}, if\ x \neq 0 \\ 0, if\ x = 0 \end{cases}$

f(x) = |x|/x can also be defined as,

$\frac{x}{x} = 1$ If $if\ x > 0\ and\ \frac{-x}{x} = -1$ if $x < 0$

$\Rightarrow f(x) = 1$ if $x > 0, f(x) = -1$ if $x < 0$ and $f(x) = 0$ if x =0

We get that, domain of $f\ (x)$ is R as $f\ (x)$ is defined $x > 0$ and $x = 0$.

For all $x > 0, f\ (x) = 1$ is a constant function and continuous

For all $x < 0, f\ (x) = -1$ is a constant function and continuous

Therefore $f\ (x)$ is continuous on R – {0}.

Now

left hand limit $= \lim_{x \to 0^-} f\ (x) = \lim_{x \to 0^-} (-1) = -1$

Right hand limit $= \lim_{x \to 0^+} f\ (x) = \lim_{x \to 0^+} (1) = 1$

As, $\lim_{x \to 0^-} f\ (x) \neq \lim_{x \to 0^+} f(x)$

therefore $\lim_{x \to 0^+} f(x)$ Does not exit and f(x) is discontinuous at only x = 0.

Find all points of discontinuity of *f*. where *f* is defined by:

Question 9

$$f\ (x) = \begin{cases} \frac{x}{|x|}, & if\ x < 0 \\ -1, & if\ x \geq 0 \end{cases}$$

Solution:

Given function is

$$f\ (x) = \begin{cases} \frac{x}{|x|}, & if\ x < 0 \\ -1, & if\ x \geq 0 \end{cases}$$

At $x = 0$ L. HL. $= \lim_{x \to 0^+} \frac{x}{|x|} = -1$ and f (0) $= -1$

R. HL. $= \lim_{x \to 0^+} f\ (x) = -1$

As, L. H. L. = R. H. L. $f\ (0)$

Therefore, $f\ (x)$ is continuous function?

Now

for $x = c < 0\ \lim_{x \to c^-} \frac{x}{|x|} = -1 = f\ (c)$

Therefore, $\lim_{x \to c^-} = f\ (x)$

Therefore, $f(x)$is a continuous at $x = c < 0$
Now for x $= c < 0$ $\lim_{x \to c^+} f(x) = 1 = f(c)$
Therefore, $f(x)$ is a continuous at all $x = c > 0$
Answer: The function is continuous at all points of its domain

Find all points of discontinuity of f. Where f is defined by:

Question 10

$$f(x) = \begin{cases} x + 1 \text{ if } x \geq 1 \\ x^2 +, \text{if } x < 1 \end{cases}$$

Solution:

$$f(x) = \begin{cases} x + 1, \text{if } x \geq 1 \\ x^2 +, \text{if } x < 1 \end{cases}$$

We know that f(x) being polynomial is continuous for $x \geq 1$ and $x < 1$ for all $x \in$ R.
Check continuity at x = 1
$\text{R. H. L.} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x + 1) = \lim_{h \to 0} (1 + h + 1) = 2$
$\text{L. H. L.} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} (x^2 + 1) = \lim_{h \to 0} ((1 - h)^2 + 1) = 2$
And $f(1) = 2$
As, L.H.L. = R.HL. = $f(1)$
Therefore f(x) is a continuous at x = 1 and x ∈ R.
Hence, $f(x)$ has no point of discontinuity

Find all points of discontinuity of f where f is defined by:

Question 11

$$f(x) = \begin{cases} x^3 - 3, if\ x \leq 2 \\ x^2 + 1, if\ x > 2 \end{cases}$$

Solution:

$$f(x) = \begin{cases} x^3 - 3, if\ x \leq 2 \\ x^2 + 1, if\ x > 2 \end{cases}$$

At $\text{x} = 2$ $\text{L. H. L.} = \lim_{x \to 2^-} (x^3 - 3) = 8 - 3 = 5$
$\text{R. H. L.} = \lim_{x \to 2^+} (x^2 + 1) = 4 + 1 = 5$
$f(2) = 2^3 - 3 = 8 - 3 = 5$
As, L.H.L = R.H.L. $f(2)$
Therefore, $f(x)$ is a continuous at $x = 2$
Now, for $(x^2 + 1) = c^2 + 1 = f(c)$ and

therefore $\lim_{x \to c^-} = f(x)$
This implies, $f(x)$ is a continuous for all x ∈ R.
Hence the function has no point of discontinuity

Find all points of discontinuity of *f* where *f* is defined by:

Question 12

$$f(x)\begin{cases} x^{10} - 1, & if\ x \leq 1 \\ x^2. & \text{if } x > 1 \end{cases}$$

Solution:

$$f(x)\begin{cases} x^{10} - 1, & if\ x \leq 1 \\ x^2. & \text{if } x > 1 \end{cases}$$

At x $= 1$ L. H. L. $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^{10} - 1) = 0$
R. H. L. $= \lim_{x \to 1^+} f(x) \lim_{x \to 1^+} (x^2) = 1$
$f(1) = 1^{10} - 1 = 0$
As, L.H.L ≠ R.H.L.
Therefore, $f(x)$ is a continuous at? $x = 1$
Now, for $x = c < 1 \lim_{x \to c}(x^{10} + 1) = c^{10} - 1 = f(c) and\ for\ x = c > 1 \lim_{x \to c}(x^2) = c^2 = f(1)$
Therefore, $f(x)$ is a continuous for all $x \in R - \{1\}$
Hence for all given function x = 1 is a point of discontinuity

Question 13

Is the function defined by $f(x) = \begin{cases} x + 5. \text{if } x \leq 1 \\ x - 5. \text{if } x > 11 \end{cases}$ a continuous function?

Solution:

Given function is $f(x) = \begin{cases} x + 5. & \text{if } x \leq 1 \\ x - 5. & \text{if } x > 1 \end{cases}$

At $x = 1$L. H. L. $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} (x + 5) = 6$
R. H. L. $= \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x - 5) = -4$
As, L.H.L. ≠ R.H.L.
Therefore, $f(x)$ is discontinuous at x $= 1$
Now, for $x = c < 1$
$\lim_{x \to c}(x + 5) = c + 5 = f(c)$ and
$for\ x = c > 1 \lim_{x \to c}(x - 5) = c - 5 = f(c)$

Therefore, f(x) is a continuous for all $x \in R - \{1\}$
Hence f(x) is not a continuous function.

Discuss the continuity of the function f, where f is defined by:

Question 14

$$f(x)\begin{cases} 3, \text{if } 0 \le x \le 1 \\ 4, \text{if } 1 < x < 3 \\ 5, \text{if } 3 \le x \le 10 \end{cases}$$

Solution:

$$f(x)\begin{cases} 3, \text{if } 0 \le x \le 1 \\ 4, \text{if } 1 < x < 3 \\ 5, \text{if } 3 \le x \le 10 \end{cases}$$

In interval, $0 \le x \le 1, f(x) = 3$
Therefore f is continuous in this interval.
At x = 1
$\text{L.H.L.} = \lim_{x \to 1^-} f(x) = 3 \text{ and } R.H.L. = \lim_{x \to 1^-} f(x) = 4$
As, L.H.L. ≠ R.H.L.
Therefore f (x) is discontinuous at x = 1
At $x = 3. \text{L.H.L.} = \lim_{x \to 3^-} f(x)$ and $\text{R.H.L.} = \lim_{x \to 3^+} f(x) = 5$
As L.H.L ≠ R.H.L.
Therefore, f(x) is discontinuous at x = 3
Hence, f is discontinuous at x = 2 and x = 3

Discuss the continuity of the function f, where f is defined by

Question 15

$$f(x)\begin{cases} 2x, if\ x < 0 \\ 0, if\ 0 \le x \le 1 \\ 4x, if\ x > 1 \end{cases}$$

Solution:

$$f(x)\begin{cases} 2x, if\ x < 0 \\ 0, if\ 0 \le x \le 1 \\ 4x, if\ x > 1 \end{cases}$$

$\text{L.H.L.} = \lim_{x \to 0^-} 2x = 0 \text{ and } \text{R.H.L.} = \lim_{x \to 0^+} (0) = 0$

As, L.H.L. = R.H.L.
Therefore, f(x) is continuous at x = 0
$As, L.H.L. = \lim_{x\to 1^-}(0) = 0$ and $R.H.L. = \lim_{x\to 1^+}(4x) = 4$
As, L.H.L. ≠ R.H.L.
Therefore, f(x) is discontinuous at x = 1.
When x<0,
f(x) is a polynomial function and is continuous for all x < 0.
When $x > 1. f(x) = 4x$
It is being a polynomial function is continuous for all$x > 1$
Hence, x = 1 is a point of discontinuity.
Discuss the continuity of the function f, where f is defined by

Question 16

$$\boldsymbol{f(x) = \begin{cases} -2, \text{ if } x \leq -1 \\ 2x, \text{ if } -1 < x \leq 1 \\ 2, \text{ if } x > 1 \end{cases}}$$

Solution:

$$f(x) = \begin{cases} -2, \text{ if } x \leq -1 \\ 2x, \text{ if} -1 < x \leq 1 \\ 2, \text{ if } x > 1 \end{cases}$$

At x = -1,
$L.H.L. = \lim_{x\to 1^-} f(x) = -2$ and $R.H.L. = \lim_{x\to 10^+} f(x) = -2$
As, L.H.L. = R.H.L.
Therefore, $f(x)$ is continuous at$x = -1$
At x = 1,
$L.H.L. = \lim_{x\to 1^-} f(x) = 2$ and $R.H.L. = \lim_{x\to 1^+} f(x) = 2$
As, L.H.L. = R.H.L.
Therefore, f(x) is continuous at x = 1.

Question 17

Find the relationship between a and b so that the function f defined by $\boldsymbol{f(x) \begin{cases} ax + 1. \text{ if } x \leq 3 \\ bx + 1. \text{ if } x > 3 \end{cases}}$ is continuous at x = 3

Solution:

$$f(x) \begin{cases} ax + 1. \text{ if } x \leq 3 \\ bx + 1. \text{ if } x > 3 \end{cases}$$

Check continuity at x = 3.

$\lim_{x\to3^+} f(x) = \lim_{x\to3^+}(ax+1) = \lim_{h\to0}\{a(3-h)+1\} = \lim_{h\to0}(3a-ah+1) = 3a+1$

$\lim_{x\to3^+} f(x) = \lim_{x\to3^+}(bx+3) = \lim_{h\to0}\{b(3-h)+3\} = \lim_{h\to0}(3a-bh+3) = 3b+3$

Also $f(3) = 3a+1$

$therefore, \lim_{x\to3^-} f(x) = \lim_{x\to3^+} f(x) = f(3)$

$\Rightarrow 3b+3 = 3a+1$

$\Rightarrow a-b = \frac{2}{3}$

For what value of λ is the function defined by

Question 18

$$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \le 0 \\ 4x+1. & \text{if } x > 0 \end{cases}$$

Continuous at x = 0? What about continuity at x = 1?

Solution:

Since $f(x)$ is continuous at $x = 0$

Therefore

L.H.L

$\lim_{x\to0^-} f(x) = f(0) = \lambda(x^2 - 2x) = \lambda(0-0) = 0$

R.H.L

$\lim_{x\to0^-} f(x) = f(x) = f(0) = 4x+1 = 4 \times 0 + 1 = 1$

Here, L.H.L. ≠ R.H.L.

This implies 0 = 1, which is not possible.

Again, f(x) is continuous at x = 1.

Therefore,

$\lim_{x\to0^-} f(x) = f(-1) = \lambda(x^2-2x) = \lambda(1-2) = 3\lambda$

$And \lim_{x\to0^-} f(x) = f(1) = 4x+1 = 4x+4 \times 1+1 = 5$

Let us say, L.H.L. = R.H.L.

$\Rightarrow 3\lambda = 5$

$\Rightarrow \lambda = \frac{5}{3}$

The value of is 3/5

Question 19

Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral points. Here$[x]$ denotes the greatest integer less than or equal to

Solution:

For any real number, x,
$[x]$ Denotes the fractional part or decimal part of x.
For example:
[2.35] = 0.35
[−5.45] = 0.45
[2] = 0
[−5] = 0
The function g: R -> R defined by $g\ (x) =\ x - [x]\ \forall\ x\ \in\ \infty$ is called the fractional part function. The domain of the fractional part function is the set R of all real numbers, and [0, 1) is the range of the set.
So, given function is discontinuous function.

Question 20

Is the function $f\ (x) =\ x^2 -\ sin\ x + 5$ continues at $x =\ \pi$?

Solution:

Given function is $f\ (x) =\ x^2 -\ sin\ x + 5$
$\text{L. H. L.} = \lim_{x \to \pi^+} (x^2 - \sin x + 5) = \lim_{x \to \pi^-} [(\pi - h)^2 - \sin(\pi - h) +\ 5] =\ \pi^2 +\ 5$
$\text{R. H. L.} = \lim_{x \to \pi^+} (x^2 - \sin x + 5) = \lim_{x \to \pi^-} [(\pi - h)^2 - \sin(\pi + h) +\ 5] =\ \pi^2 +\ 5$
And $f\ (\pi) =\ \pi^2 - \sin \pi + 5 =\ \pi^2 +\ 5$
Since L.H.L = R.H.L. = $f\ (\pi)$
Therefore, f is continuous at $x =\ \pi$

Question 21

Discuss the continuity of the following functions:
a) $f\ (x) = sin\ x + cos\ x$
b) $f\ (x) =\ sin\ x - cos\ x$
c) $f\ (x) =\ sin\ x. cos\ x$

Solution:

a) Let ``a” be an arbitrary real number then
$\lim_{x \to a^+} f\ (x) \Rightarrow \lim_{h \to 0} (a + h)$
Now
$\lim_{x \to a^+} f\ (a + h) = \lim_{h \to 0} (a + h) + \cos(a + h)$
$= \lim_{h \to 0} (\sin a \cos 0 + coas\ a \sin h + cos\ a \cos h - \sin a \sin h)$
$= \sin a \cos 0 + \cos a \sin 0 + \cos a \cos 0 - \sin a \sin 0$
{As cos 0 = 1 and sin 0 = 0}
= sin a + cos a = f (a)

Similarly,

$\lim_{x \to a^-} f(x) = f(a)$

$\lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a^+} f(x)$

Therefore, f (x) is continuous at x = a

As, ``a" is an arbitrary real number therefore, $f(x) = \sin x + \cos x$ is continuous

b) Let ``a" is arbitrary real number then $\lim_{x \to a^+} f(x) \Rightarrow \lim_{h \to 0} f(a+h)$

Now

$\lim_{h \to 0} f(a+h) = \lim_{h \to 0} \sin(a+h) - \cos(a-h)$

$\Rightarrow \lim_{h \to 0} (\sin a \cos h + \cos a \sin h - \cos a \cos h - \sin a \sin h)$

$= \sin a \cos 0 + \cos a \sin 0 - \cos 0 - \sin a \sin 0$

$= \sin a + 0 - \cos a - 0$

$= \sin a - coa\ a = f(a)$

similarly,

$\lim_{x \to a^-} f(x) = f(a)$

$\lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a^+} f(x)$

Therefore, f (x) is continuous at x = a

Since ``a" is a arbitrary real number, therefore, $f(x) = \sin x - \cos x$ is continuous

c) Let ``a" be an arbitrary real number then $\lim_{x \to a^+} f(x) \Rightarrow \lim_{h \to 0} f(a+h)$

$Now\ \lim_{h \to 0} f(a+h) = \lim_{h \to 0} \sin(a+h).\cos(a+h)$

$= \lim_{h \to 0} (\sin a \cos h + \cos a \sin h)\ (\cos a \cos h - \sin a \sin h)$

$= (\sin a \cos 0 + \cos a \sin 0)\ (\cos a \cos 0 - \sin a \sin 0)$

$= (\sin a + 0)\ (\cos a - 0)$'

$= \sin a . \cos a = f(a)$

similarly $\lim_{x \to a^-} f(x) = f(a)$

$\lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a^+} f(x)$

Therefore, f(x) is continuous at x = a.

Since, "a" is an arbitrary real number, $f(x) = \sin x \cos x$ therefore, is continuous.

Question 22

Discuss the continuity of cosine, cosecant, secant and cotangent functions.

Solution:

Continuity of cosine:

Let say "a" be an arbitrary real number then

$\lim_{x \to a^+} f(x) \Rightarrow \lim_{z \to a^+} \cos x \Rightarrow \lim_{h \to 0} \cos(a+h)$

which implies, $\lim_{h \to 0} (\cos a \cos h - \sin a \sin h)$

$= \cos a \lim_{h \to 0} \cos h - \sin a \lim_{h \to 0} \sin h$

$= \cos a \times 1 - \sin a \times 0 = \cos a = f(a)$

$\lim_{x \to a} f(x) = f(a)$ for all a $\in$ R

Therefore, f(x) is continuous at x = a.

Since, "a" is an arbitrary real number, therefore, is continuous.

Continuity of cosecant:

Let say "a" be an arbitrary real number then

$f(x) = cosec\, x = \frac{1}{\sin x}$ and

domain $x = R - (x\,\pi), x \in |$

$\Rightarrow \lim_{x \to a} \frac{1}{\sin x} = \frac{1}{\lim_{h \to 0} \sin(a+h)}$

$= \frac{1}{\lim_{h \to 0} (\sin a \cos h + \cos a \sin h)}$

$= \frac{1}{\sin a \cos 0 + \cos a \sin 0}$

$= \frac{1}{\sin a\,(1) + \cos a (0)}$

$= \frac{1}{\sin a} = f(a)$

Since ``a" is an arbitrary real number, therefore, $f(x) = coses\ x$ is continuous.

Continuity of secant:

Let say ``a" be an arbitrary real number then

$f(x) = \sec x = \frac{1}{\cos x}$ and domain$|\ x = \mathrm{R} - (2x+1)\frac{\pi}{2}, x \in |$

$\Rightarrow \lim_{x \to a} \frac{1}{\cos x} = \frac{1}{\lim_{h \to 0} \cos(a+h)}$

$= \frac{1}{\lim_{h \to 0} (\cos a \cos h - \sin a \sin h)}$

$= \frac{1}{\cos a \cos 0 - \sin a \sin 0}$

$= \frac{1}{\cos a (1) - \sin a\,(0)} = \frac{1}{\cos a} = f(a)$

Therefore , f(x) s continuous at x = a

Since ``a" is an arbitrary real number, therefore, $f(x) = \sec x$ is continuous

Continuity of cotangent

Let say ``a" be an arbitrary real number then

$f(x) = \cot x \frac{1}{\tan x}$ and domain $x\ R - (x\pi), x\ \pi|$

$\Rightarrow \lim_{x \to 0} \frac{1}{\tan x} = \frac{1}{\lim_{h \to 0} \tan(a+h)}$

$$= \frac{1}{\lim_{x\to 0}\left(\frac{\tan a + \tan h}{1 - \tan a \tan h}\right)} = \frac{1}{\frac{\tan a + 0}{1 - \tan a \tan 0}}$$

$$= \frac{1-0}{\tan a} = \frac{1}{\tan a} = f(a)$$

Therefore f(x), I continuous at a = a

Since ``a” is an arbitrary real number $f(x) = \cot$ is continuous

Question 23

Find all points of discontinuity of f where ,

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x < 0 \\ x + 1 & \text{if } x \geq 0 \end{cases}$$

Solution:

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x < 0 \\ x + 1 & \text{if } x \geq 0 \end{cases}$$

At x = 0

$\text{L.H.L.} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin(-h)}{-h} = 1$

$\text{R.H.L.} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x+1) = 0 + 1 = 1$

$f(0) = 1$

Therefore, f is continuous at $x = 0$

When $x < 0$. $\sin x$ and x are continuous, then $\frac{\sin x}{x}$ is also continuous.

When $x > 0$. $f(x) = x + 1$ is a polynomial, then f is continuous.

Therefore, f is continuous at any point.

Question 24

Determine if f defined by

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Is a continuous function.

Solution:

Given function is:

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$\lim_{x \to 0} f(x) = \lim_{z \to 0} x^2 \sin\frac{1}{x}$

As we know, sin (1/x) lies between -1 and 1, so the value of sin 1/x is any integer, say m, we have

$\lim_{x\ 0} f(x) = \lim_{z \to 0} x^2 \sin\frac{1}{x}$

= 0 x m

= 0

And, f(0) = 0

Since, $\lim_{x \to 0} f(x) = f(0)$, therefore, the function is continuous at

Question 25

Examine the continuity of f, where f is defined by

$$f(x) = \begin{cases} \sin x - \cos x \ \ if\ x \neq 0 \\ -1, if\ x = 0 \end{cases}$$

Solution:

$$f(x) = \begin{cases} \sin x - \cos x \ \text{ if } x \neq 0 \\ -1, \text{if } x = 0 \end{cases}$$

Find left hand and right hand limits at x = 0

$At\ x = 0, \text{L.H.L.} = \lim_{x \to 0} f(x) = \lim_{h \to 0} (0 - h) = \lim_{h \to 0} f(-h)$

$\Rightarrow \lim_{h \to 0} \sin(-h) + cos\ (-h) = \lim_{h \to 0} (-\sin h + \cos h) = -0 + 1 = 1$

$\text{R.H.L.} = \lim_{x \to 0} f(x) = \lim_{h \to 0} (0 + h) = \lim_{h \to 0} f(h)$

$\Rightarrow \lim_{h \to 0} \sin(h) + cos\ (h) = \lim_{h \to 0} (\sin h + \cos h) = -0 + 1 = 1$

And f (0) = 1

Therefore, $\lim_{h \to 0^-} f(x) = \lim_{h \to 0^+} f(x) \neq f(0)$

Therefore, f (x) I discontinues at x = 0

Find the values of k so that the function f is continuous at the indicated point in exercise 26 to 29.

Question 26

$$f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, \text{if } x \neq \frac{\pi}{2} \\ 3, \text{if } x = \frac{\pi}{2}\ at\ x = \frac{\pi}{2} \end{cases}$$

Solution:

Given function is

$$f(x) = \begin{cases} \frac{k\cos x}{\pi - 2x}, \text{if } x \neq \frac{\pi}{2} \\ 3, \text{if } x = \frac{\pi}{2} \; at \; x = \frac{\pi}{2} \end{cases}$$

$$\lim_{x \to \frac{\pi}{2}} f(x) = \lim_{x \to \frac{\pi}{2}} \frac{k\cos x}{\pi - 2x}$$

So, $x \to \frac{\pi}{2}$

This implies $x \neq \frac{\pi}{2}$

Putting $x = \frac{\pi}{2} + h$ where $h \to 0$

$$= \lim_{h \to 0} k\cos \frac{\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)}$$

$$= \lim_{h \to 0} \frac{-k\sin h}{\pi - \pi - 2h}$$

$$= \lim_{h \to 0} \frac{-k\sin h}{-2h}$$

$$= \frac{k}{2} \times \lim_{h \to 0} \frac{\sin h}{h}$$

$$= \frac{k}{2} \ldots\ldots\ldots\ldots (1)$$

And $f\left(\frac{\pi}{2}\right) = 3 \ldots\ldots\ldots\ldots. (2)$

$f(x) = 3 \; when \; x = \frac{\pi}{2}$ [Given]

As we know, f(x) is continuous at x= $\pi/2$

$$\lim_{x \to \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$$

From equation (1) and equation (2), we have

$\frac{k}{2} = 3$

K = 6

Therefore the value of k is 6.

Question 27

$$\boldsymbol{f(x) = \begin{cases} kx^2 \text{if } x \le 2 \\ 3, \text{if } x > 2 \; at \; x = 2' \end{cases}}$$

Solution:

Given function is

$$f(x) = \begin{cases} kx^2 \text{if } x \le 2 \\ 3, \text{if } x > 2 \end{cases}$$

$\lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h) = 3$
$\lim_{x\to 2^-} f(x)\ and\ f(2) = 3$
$k \times 2^2 = 3$
This implies, $k = \frac{3}{4}$
when k = 3/4 then $\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0}\frac{3}{4}(2-h)^2 = 3$
Therefore, $f(x)$ is continuous at $x = 2$ when $k = \frac{3}{4}$.

Question 28

$$f(x) = \begin{cases} k\,x + 1.\text{ if } x \le \pi \\ \cos x \text{ if } x > \pi\ at\ x = \pi \end{cases}$$

Solution:

$$f(x) = \begin{cases} k\,x + 1.\text{ if } x \le \pi \\ \cos x \text{ if } x > \pi\ at\ x = \pi \end{cases}$$
$\lim_{x\to \pi^-} f(x) = \lim_{h\to 0} f(\pi+h) = \lim_{h\to 0}(\pi+h) = -\cos h = -cos0 = -1$
and $\lim_{x\to \pi^-} f(x) = \lim_{h\to 0} f(\pi-h) = \lim_{h\to 0}(\pi-h) = -\cos h = -cos0 = -1$
Again
$\lim_{x\to \pi} f(x) = \lim_{h\to 0}(k\,\pi+1)$
Again given function is continuous at $x = \pi$ we have
$\lim_{x\to \pi} f(x) = \lim_{x\to 0} f(x) = \lim_{z\to 0} f(x)$
$\Rightarrow k\pi + 1 = 1$
$\Rightarrow k\pi = -2$
$\Rightarrow k = \dfrac{-2}{\pi}$
The value of k is -2/π

Question 29

$$f(x) = \begin{cases} kx + 1, \text{ if } x \le 5 \\ 3x - 5, \text{ if } x > 5\ at\ x = 5 \end{cases}$$

Solution:

Given function is
$$f(x) = \begin{cases} kx + 1, if\ x \le 5 \\ 3x - 5, if\ x > 5 \end{cases}$$
When x < 5, $f(x) = kx + 1$:A polynomial is continuous at each point x < 5.
When x < 5, $f(x) = 3kx - 5$: A polynomial is continuous at each point x < 5.
Now $f(5) = 5k + 1 = 3(5+h) - 5$

$\lim_{x \to 5^+} f(x) = \lim_{h \to 0} f(5+h) = 15 + 3h - 5 \ldots \ldots \ldots \ldots .. (1)$

$= 10 + 3h = 10 + 3 \times 0 = 10$

$\lim_{x \to -} f(x) = \lim_{h \to 0} f(5-h) = k(5-h) + 1 = 5k - nk + 1 = 5k + 1 \ldots \ldots \ldots \ldots .. (2)$

Since function is continuous, therefore, the equation is equal; equate both the equation and find the value of k,

$10 = 5k + 1$

$5k = 9$

$k = \frac{9}{5}$

Question 30

Find the values of a and b such that the function defined by

$$f(x) = \begin{cases} 5, if\ x \le 2 \\ ax + b, if\ 2 < x\ 10 \\ 21, if\ x \ge 10 \end{cases}$$

Is a continuous functions

Solution:

$$f(x) = \begin{cases} 5, if\ x \le 2 \\ ax + b, if\ 2 < x\ 10 \\ 21, if\ x \ge 10 \end{cases}$$

For x < 2; function is f(x) =5; which is a constant.

Function is continuous.

For 2 < x < 10; function f(x) = ax + b; a polynomial.

Function is continuous.

For x > 10; function is f(x) =21; which is a constant.

Function is continuous.

Now, for continuity at $x = 2$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$

$\Rightarrow \lim_{h \to 0} (5) = \lim_{h \to 0} \{a(2-h) + b\} = 5$

$\Rightarrow 2a + b = 5 \ldots \ldots \ldots \ldots . (1)$

For continuiuty $at\ x = 10.$ $\lim_{x \to 10^-} f(x) = \lim_{x \to 10^+} f(x) = f(10)$

$\Rightarrow \lim_{h \to 0} (21) = \lim_{h \to 0} \{a(10-h) + b\} = 5$

$\Rightarrow 10a + b = 21 \ldots \ldots \ldots \ldots . (2)$

Solving equation (1) and equation (2), we get

a = 2 and b = 1.

Question 31

Show that the function defined by f(x) = cos (x2) is a continuous function.

Solution:

Given function is:
f(x) = cos (x2)
Let g(x) = cos x and h(x) = x^2, then
goh(x) = g(h(x))
= g(x^2)
= cos (x^2)
=f(x)
This implies, goh(x) = f(x)
Now,
g(x) = cos x is continuous and
h(x) = x^2 (a polynomial)
[We know that, if two functions are continuous then their composition is also continuous]
So, goh(x) is also continuous.
Thus f(x) is continuous.

Question 32

Show that the function defined by f(x) = |cos x| is a continuous function.

Solution:

Given function is
f(x) = |cos x|
f(x) is a real and finite for all x ∈ R and Domain of f(x) is R.
Let $\mathbf{g}(x) = \cos x$ and $h\,(x) = \ |x|$
Here, g(x) and h(x) are cosine function and modulus functions are continuous for all real x.
Now,(**g**oh) $x = \ g\{h\,(x) = \ g|(x)| = cos\,|x|\}$ is also is continuous being a composite function of two continuous functions, but not equal to f(x).
Again, $(hog)x = h\,\{g(x)\} = h\,(cos\,x) = |cos\,x| = f\,(x)$[Using given]
Therefore $f\,(x) = |\cos x| = (hog)x$ is composite function of two continuous function is continuous.

Question 33

Examine that sin $|x|$ is a continuous function.

Solution:
Let $f\,(x) = \ |x|\ and\ g\,(x) = \ sin\,|x|$then
$(gof)x -= \ g\,\{f\,(x)\} = g\,(|x|) = sin\,|x|$
Now, f and g are continuous, so their composite, (**g** of) is also continuous.
Therefore, sin$|x|$ is continuous.

Question 34

Find all points of discontinuity of f defined by f (x) = $|x| - |x+1|$

Solution:

Given function is $f\ (x) = |x| - |x+1|$
When x < -1: $f\ (x) = -x - \{-(x+1)\} = -x + x + 1 = 1$
When $-1 \leq x < 0; f\ (x) = -x - (x+1) = -2x - 1$
When $x \geq 0.; f\ (x) = x - (x+1) = 1$
So, we have a function as:

$$f\ (x) \begin{cases} 1, if\ x < -1 \\ -2x - 1, if - 1 \leq x < 0 \\ -1, if\ x \geq 0 \end{cases}$$

Check the continuity at x = -1, x = 0
At $x = -1$, L. H. L. $\lim_{x \to 1^-} f\ (x) = \lim_{Z \to 1^-} 1 = 1$
$R.H.L. = \lim_{x \to 1^-} f\ (x) = \lim_{x \to 1^-} (-2x -) = 1$
And $f\ (-2) = 2 \times 2 - 1 = 1$
Therefore at $x = -1, f\ (x)$ is continuous
At $x = 0\ L.H.L. = \lim_{x \to 0^-} f\ (x) = \lim_{x \to 0^-} (-2x - 1) = -1\ and\ R.H.L. = \lim_{x \to 0^-} f\ (x) = \lim_{x \to 0^-} (-1) = -1$
Therefore, at x = 0, f(x) is continuous
There is no point of discontinuity.

Exercise 5.2

Question 1

Differentiate the functions with respect to x in Exercise 1 to 8.
$\boldsymbol{sin\ (x^2 + 5)}$

Solution:

Let $y = sin\ (x^2 + 5)$
Apply derivative both the side with respect to x.

$$\frac{dy}{dx} = \cos(x^2 + 5)\ \frac{d}{dx}\ (x^2 + 5)$$

$= \cos(x^2 + 5)\ (2x + 0)$
$= 2x \cos(x^2 + 5)$

Question 2

$\cos(\sin x)$

Solution:

let $y = \cos(\sin x)$
Apply derivative both the sides with respect to x.
$\frac{dy}{dx}$= - $\sin(\sin x)\frac{d}{dx}\sin x$
$= -\sin(\sin x)\cos x$

Question 3

$\sin(a x + b)$

Solution:

Let y $\sin(a x + b)$
Apply derivative both the sides with respect to x.
$\frac{dy}{dx} = cos\,(ax + b)\frac{d}{dx}\,(ax + b)$
$= cos\,(x + b)(a + b) = a\,cos\,(ax + b)$

Question 4

$sec\left(tan\sqrt{x}\right)$

Solution:

Let $y = \sec\left(\tan\sqrt{x}\right)$
Apply derivative both the sides with respect to x.
$\frac{dy}{dx} = \sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right) sec^2\sqrt{x}\,\frac{d}{dx}\sqrt{x}$
$= \sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right) sec^2\sqrt{x}\frac{1}{2}x^{\frac{1}{2}-1}$
$= \sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right) sec^2\sqrt{x}\frac{1}{2\sqrt{x}}$

Question 5

$\frac{\sin(ax + b)}{\cos(cx + d)}$

Solution:

$Let\ \frac{\sin(ax+b)}{\cos(cx+d)}$

Using quotient rule.

$$\frac{dy}{dx} = \frac{\cos(cx+d)\frac{d}{dx}\sin(ax+b)) - \sin(ax+b)\frac{d}{dx}\cos(cx+d)}{cos^2\,(cx+d)}$$

$$= \frac{\cos(cx+d)\cos(ax+b)\frac{d}{dx}\,(ax+b) - \sin(ax+b)\{(cx+b)\}\,\frac{d}{dx}\,(cx+d)}{cos^2\,(cx+d)}$$

$$= \frac{\cos(cx+d)\cos(ax+b)(a) + \sin(ax+b)\sin(cx+d)(c)}{cos^2\,(cx+d)}$$

Question 6

$cos\ x^3\ sin^2\ (x^5)$

Solution:

Let $y = \ cos\ x^3 . sin^2(x^5)$

Apply derivative both the sides with respect to x.

$$\frac{d}{dx} = \ cos\ x^3\ \frac{d}{dx}\ sin^2\ (x^5) + \ sin^2\ (x^5)\frac{d}{dx}\ cos\ x^3$$

$$= \ \cos x^3\ .2\sin(x^5)\frac{d}{dx}\sin(x^5) + \ sin^2\ (x^5)\ (-\sin x^3)\frac{d}{dx}\ x^3$$

$$= \cos x^3\ .2\sin(x^5)\cos(x^5)(5x^4) - sin^2\ (x^5)\sin x^3 . 3x^2$$

$$= 10x^4\cos x^3\sin(x^5)\cos(x^5) - 3x^2\ sin^2(x^5)\sin x^3$$

Question 7

$\sqrt[2]{cot\ (x^2)}$

Solution:

Let $y = \sqrt[2]{cot\ (x^2)}$

Apply derivative both the sides with respect to x.

$$\frac{dy}{dx} = 2.\frac{1}{2}\{cot(x^2)\}^{\frac{-1}{2}} . \frac{d}{dx}\ cot\ (x^2)$$

$$= \frac{1}{\sqrt{\cot(x^2)}} . \{-cosec\ (x^2)\}\frac{d}{dx}x^2$$

$$= \frac{1}{\sqrt{\cot(x^2)}} . \{-cosec\ (x^2)\}\ (2x)$$

$= \frac{-2x\, coosec(x^2)}{\sqrt{\cot(x^2)}}$

Question 8

$\boldsymbol{cos\ (\sqrt{x})}$

Solution:

Let y = $cos\ (\sqrt{x})$
Apply derivative both the sides with respect to x.
$\frac{dy}{dx} = -sin\ \sqrt{x}\ \frac{d}{dx}\sqrt{x}$
$= -sin\sqrt{x}\frac{1}{2}\ (x)^{\frac{-1}{2}} = \frac{-sin\sqrt{x}}{2\sqrt{x}}$

Question 9

Prove that the function f given by $f\ (x) = |x-1|, x \in R$ is not differentiable at x = 1

Solution:

Given function: $f\ (x) = |x-1|$
$f\ (1) = |x-1| = 0$
Right hand limit: $f\ (1) = \lim_{h\to 0}\frac{f\ (1+h) - f\ (1)}{h}$
$= \lim_{h\to 0}\frac{|1+h-1|-0}{h}$
$= \lim_{h\to 0}\frac{|h|}{h} = \lim_{h\to 0}\frac{h}{h} = 1$
And left hand limit
$f\ (1) = \lim_{h\to 0}\frac{f\ (1-h) - f\ (1)}{-h}$
$= \lim_{h\to 0}\frac{|1+h-1|-0}{-h}$
$= \lim_{h\to 0.}\frac{|-h|}{-h}$
$= \lim_{h\to 0}\frac{-h}{h} = -1$
Right hand limit ≠ Left hand limit
Therefore, f(x) is not differentiable at x = 1

Question 10

Prove that the greatest integer function defined by $f\ (x) = [x], 0 < x < 3$ is not differentiable at x =1 and x = 2

Solution:

Give function is

$f\ (x) = [x], 0 < x < 3$

Right hand limit:

$$f'(1) = \lim_{h \to 0} \frac{f\ (1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{|1+h| - 1}{h}$$

$$= \lim_{h \to 0} \frac{1-1}{h}$$

$$= \lim_{h \to 0} \frac{0}{h} = 0$$

and left hand limit

$$f'(1) = \lim_{h \to 0} \frac{f\ (1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{|1-h| - 1}{-h}$$

$$= \lim_{h \to 0} \frac{0-1}{-h} = \infty$$

Right hand limit ≠ Left hand limit

Therefore $f\ (x) = [x]$ is not differentiable at x=1.

In same way, $f\ (x) = [x]$is not differentiable at x = 2

Exercise 5.3

Find $\frac{dy}{dx}$ in the following exercise 1 to 5.

Question 1

$$2x + 3y = \sin x$$

Solution:

Given function is $2x + 3y = \sin x$

Derivate function with respect to x, we have

$$\frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx}\sin x$$

$2+3\frac{dy}{dx}=\cos x$

$3\frac{dy}{dx}=x-2$

$\frac{dy}{dx}=\frac{\cos x-2}{3}$

Question 2

$\mathbf{2x+3y=\sin y}$

Solution:

Give function is 2x +3y = sin y

Derivate function with respect to x, we have

$\frac{d}{dx}(2x)+\frac{d}{dx}(3y)=\frac{d}{dx}\sin y$

$=2+3\frac{dy}{dx}=\cos \mathrm{y}\frac{\mathrm{dy}}{\mathrm{dx}}$

$=\frac{dy}{dx}(\cos \mathrm{y}-3)=-2$

$=\frac{dy}{dx}=\frac{2}{\cos \mathrm{y}-3}$

Question 3

$\boldsymbol{ax+\ by^2=\cos y}$

Solution:

Given function $ax+\ by^2\ =\cos y$

Derivate function with respect to x we have

$\frac{d}{dx}\ (ax)+\frac{d}{dx}(by^2)\ =\ \frac{d}{dx}\cos y$

$a+b.2y\ \frac{dy}{dx}=\ -\sin y\ \frac{dy}{dx}$

$2by\ \frac{dy}{dx}+\sin y\ \frac{dy}{dx}=\ -a$

$-\frac{dy}{dx}\ (2by+\sin y)=-a$

$\frac{dy}{dx}=\ \frac{-a}{2\ by+\sin y}$

Question 4

$xy + y^2 = \tan x + y$

Solution:

Given function $xy + y^2 = tan\, x + y$
Derivate function with respect to x, we have

$$\frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}\tan x + \frac{d}{dx} y$$

$$x\frac{d}{dx}y + y\frac{d}{dx}x + 2y\frac{dy}{dx} = sec^2 x + \frac{dy}{dx}$$

[Solving first time using Product Rule]

$$x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = sec^2 x + \frac{dy}{dx}$$

$$x\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{dy}{dx} = sec^2 x - y$$

$$(x + 2y - 1)\frac{dy}{dx} = sec^2 x - y$$

$$\frac{dy}{dx} = \frac{sec^2 x - y}{x + 2y - 1}$$

Question 5

$x^2 + xy + y^2 = 100$

Solution:

Give function is $x^2 + xy + y^2 = 100$
Derivate function with respect to x, we have

$$\frac{d}{dx}x^2 + \frac{d}{dx}xy + \frac{d}{dx}y^2 = \frac{dy}{dx}100$$

$$2x + \left(x\frac{d}{dx}y + y\frac{d}{dx}x\right) + 2y\frac{dy}{dx} = 0$$

$$2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$$

$$(x + 2y)\frac{dy}{dx} = -2x - y$$

Question 6

$x^3 + x^2 y + xy^2 + y^3 = 81$

Solution:

Given function is $x^3 + x^2 y + xy^2 + y^3 = 81$
Derivate function with respect to x, we have

$$\frac{d}{dx} x^3 + \frac{d}{dx} x^2y + \frac{d}{dx} xy^3 = \frac{d}{dx} 81$$

$$3x^2 + \left(x^2 \frac{dy}{dx} + y \frac{d}{dx}x^2\right) + x \frac{d}{dx}y^2 + y^2 \frac{d}{dx} x + 3y^2 \frac{dy}{dx} = 0 \text{ (using product rule)}$$

$$3x^2 + x^2 \frac{dy}{dx} + y.2x + x2y \frac{dy}{dx} + y^2.1 + 3y^2 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x^2 + 2xy + y^2) = 3x^2 - 2xy - y^2$$

$$\frac{dy}{dx} = \frac{(3x^2 + 2xy + y^2)}{x^2 + 2xy + 3y^2}$$

Question 7

$sin^2 y + cos\, x\, y = \pi$

Solution:

Given function is $sin^2 y + cos\, x\, y = \pi$
Derivate function with respect to x, we have

$$\frac{d}{dx}(\sin y)^2 + \frac{d}{dx}\cos xy = \frac{d}{dx}(\pi)$$

$$2\sin y \frac{d}{dx}\sin y - \sin xy \frac{d}{dx}(xy) = 0$$

$$2\sin y \cos y \frac{dy}{dx} - \sin xy \left(x \frac{dy}{dx} + y.1\right) = 0$$

$$\sin 2y \frac{dy}{dx} - x\sin xy \frac{dy}{dx} - y\sin xy = 0$$

$$(\sin 2y - x\sin xy)\frac{dy}{dx} = y\sin xy$$

$$\frac{dy}{dx} = \frac{y\sin xy}{\sin 2y - x\sin x\, y}$$

Question 8

$sin^2 x + cos^2 y = 1$

Solution:

Given function is $sin^2 x + cos^2 y = 1$
Derivate function with respect to x, we have

$$\frac{d}{dx}(sin\, x)^2 + \frac{d}{dx}(cos\, x)^2 = \frac{d}{dx}(1)$$

$2\sin x \frac{d}{dx}\sin x + 2\cos y \frac{d}{dx}\cos\ y = 0$

$2\ sin\ x\ cos\ x + 2\ cos\ y\ \left(-sin\ y\ \frac{dy}{dx}\right) = 0$

$\sin 2x - sin\ 2y\ \frac{dy}{dx} = 0$

$-\sin 2y\ \frac{dy}{dx} = \ -\sin 2x$

$\frac{dy}{dx} = \frac{\sin 2\ x}{\sin 2\ y}$

Question 9

$$y = \ sin^{-1}\left(\frac{2x}{1 +\ x^2}\right)$$

Solution:

Given function is

$y = \ sin^{-1}\left(\frac{2x}{1 +\ x^2}\right)$

Step 1: simplify the given function

Put x = tan θ, we have

$y = \ sin^{-1}\left(\frac{2\tan\theta}{1 +\ tan^2\ \theta}\right)$

$= \ sin^2\ (\sin\theta) = 2\theta$

Result in terms of x, we get

$y = 2\ tan^{-1}\ x$

Step 2: Derivative the function

$\frac{dy}{dx} = 2\ \frac{1}{1 +\ x^2} = \frac{2}{1 +\ x^2}$

Question 10

$$y = tan^{-1}\left(\frac{3x -\ x^3}{1 - 3x^2}\right).\frac{-1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$$

Solution:

Given function is:

$y = tan^{-1}\left(\frac{3x -\ x^3}{1 - 3x^2}\right).\frac{-1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

Step 1: simplify the given function

$$y = tan^{-1}\left(\frac{3\tan\theta - tan^3\theta}{1 - 3tan^2\theta}\right)$$

$$y = tan^{-1}(\tan 3\theta) = 3\theta$$

Step 2: Derivative the function

$$\frac{dy}{dx} = 3.\frac{1}{1+x^2} = \frac{3}{1+x^2}$$

Question 11

$$\boldsymbol{y = cos^{-1}\left(\frac{1-x^2}{1+x^2}\right). 0 < x < 1}$$

Solution:

Given function is

$$y = cos^{-1}\left(\frac{1-x^2}{1+x^2}\right). 0 < x < 1$$

Step 1: simplify the given function

Put $x \tan\theta$

$$y = cos^{-1}\left(\frac{1-tan^2\theta}{1+tan^2\theta}\right)$$

$$= cos^{-1}(\cos 2\theta) = 2\theta = 2\,tan^{-1}x$$

Step 2: Derivative the function

$$\frac{dy}{dx} = 2\,\frac{1}{1+x^2} = \frac{2}{1+x^2}$$

Question 12

$$\boldsymbol{y = sin^{-1}\left(\frac{1-x^2}{1+x^2}\right) 0 < x < 1}$$

Solution:

Given function is $y = sin^{-1}\left(\frac{1-x^2}{1+x^2}\right) 0 < x < 1$

Step 1: simplify the given function

Put $x \tan\theta$

$$y = sin^{-1}\left(\frac{1-tan^2\theta}{1+tan^2\theta}\right)$$

$$= sin^{-1}(\cos 2\theta)$$

$$= sin^{-1}\sin\left(\frac{\pi}{2} - 2\theta\right) = \frac{\pi}{2} - 2\theta$$

$= \frac{\pi}{2} - 2\,tan^{-1}\,x$

Step 2: Derivative the function

$\frac{dy}{dx} = 0 - 2\,\frac{1}{1+x^2} = \frac{-2}{1+x^2}$ (Derivative of a constant is always revert a value zero)

Question 13

$$y = cos^{-1}\left(\frac{2x}{1+x^2}\right) = -1 < x < 1$$

Solution:

Given function is $y = cos^{-1}\left(\frac{2x}{1+x^2}\right) = -1 < x < 1$

Step 1: simplify the given function

Put $x = \tan\theta$

$= cos^{-1}\left(\frac{2\tan\theta}{1+tan^2\,\theta}\right)$

$= cos^{-1}(cos2\theta)$

$= cos^{-1}\cos\left(\frac{\pi}{2} - 2\theta\right) = \frac{\pi}{2} - 2\theta$

$= \frac{\pi}{2} - 2\,tan^{-1}\,x$

Step 2: Derivative the function

$\frac{dy}{dx} = 0 - 2.\frac{1}{1+x^2} = \frac{-2}{1+x^2}$ (Derivative of a constant is zero)

Question 14

$$y = sin^{-1}\left(2x\sqrt{1-x^2}\right), \frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$$

Solution:

Given function is $y = sin^{-1}\left(2x\sqrt{1-x^2}\right), \frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

Step 1: simplify the given function

Put $x = \sin\theta$

$y = sin^{-1}\left(2\sin\theta\sqrt{1-sin^2\,\theta}\right)$

$= sin^{-1}\left(2\sin\theta\sqrt{cos^2\,\theta}\right)$

$= sin^{-1}\,(2\,sin\,\theta\,cos\,\theta)$

$= sin^{-1}\,(\sin 2\theta) = 2\theta = 2sin^{-1}x$

Step 2: Derivative the function

$\frac{dy}{dx} = 2.\frac{1}{1+x^2} = \frac{2}{1-x^2}$

Question 15

$y = sec^{-1}\left(\frac{1}{2x^2-1}\right), 0 < x < \frac{1}{\sqrt{2}}$

Solution:

Given function is $y = sec^{-1}\left(\frac{1}{2x^2-1}\right), 0 < x < \frac{1}{\sqrt{2}}$

Step 1: simplify the given function

Put $x = \cos\theta$

$y = y = sec^{-1}\left(\frac{1}{2\,cos^2\theta - 1}\right)$

$y = sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$

$= sec^{-1}(\sec 2\,\theta)$

$= 2\theta = cos^{-1} x$

Step 2: Derivative the function

$\frac{dy}{dx} = 2.\frac{-1}{1+x^2} = \frac{-2}{1-x^2}$

Exercise 5.4

Differentiate the functions with respect to x in Exercise 1 to 10

Question 1

$\frac{e^x}{\sin x}$

Solution:

Let y $= \frac{e^x}{\sin x}$

Differentiate the function with respect to x, we get,

$\frac{dy}{dx} = \frac{\sin x \frac{d}{dx} e^x - e^x \frac{d}{dx}\sin x}{sin^2 x}$

[Using quotient rule]

$= \frac{\sin x\, e^x - e^x \cos x}{sin^2 x}$

$= e^x \frac{(\sin x - \cos x)}{sin^2 x}$

Question 2

$e^{sin - x}$

Solution:

Let $y = e^{sin - x}$

Differentiate the function with respect to xx we get,

$$\frac{dy}{dx} = e^{sin - x} . \frac{d}{dx} sin^{-1}x$$

$$= e^{sin - x} . \frac{1}{\sqrt{1 - x^2}}$$

$$\left[\because \frac{d}{dx} e^{f(x)} = e^{f(x)} \frac{d}{dx} f(x)\right]$$

Question 3

e^{x^3}

Solution:

Let $y = e^{x^3} = e^{(x^3)}$

Differentiate the function with respect to x we get,

$$\frac{dy}{dx} e^{(x^3)} \frac{d}{dx} x^3$$

$$e^{(x^3)} 3x^2 = 3x^2 e^{(x^3)}$$

$$\left[\because \frac{d}{dx} e^{f(x)} = e^{f(x)} \frac{d}{dx} f(x)\right]$$

Question 4

$sin(tan^{-1} e^{-x})$

Solution:

Differentiate the function with respect to x we get,

$$\frac{dy}{dx} = cossin (tan^{-1} e^{-x}) \frac{d}{dx} (tan^{-1} e^{-x})$$

$$\left[\because \frac{d}{dx} \sin f(x) = cos f(x) \frac{d}{dx} f(x)\right]$$

$$= \cos(tan^{-1} e^{-x}) \frac{1}{1 + (e^{-x})^2} \frac{d}{dx} e^{-x}$$

$$\left[\because \frac{d}{dx} tan^{-1} f(x) = \frac{1}{(f(x))^2} \frac{d}{dx} f(x)\right]$$

$$= cos\,(tan^{-1} e^{-x}) \frac{1}{1+ e^{-2x}} e^{-x} \frac{d}{dx}(-x)$$

$$= \frac{e^{-x} cos\,(tan^{-1} e^{-x})}{1+ e^{-2x}}$$

Question 5

$\log(\cos e^x)$

Solution:

Let $y = \log(\cos e^x)$

Differentiate the function with respect to x we get,

$$\frac{dy}{dx} = \frac{1}{\cos e^x} \frac{d}{dx}(\cos e^z) \left[\because \frac{d}{x} \log f(x) = \frac{1}{f(x)} \frac{d}{dx} f(x)\right]$$

$$= \frac{1}{\cos e^z} (-\sin e^x) \frac{d}{dx} e^x \left[\because \frac{d}{x} \cos f(x) = -\sin f(x) \frac{d}{dx} f(x)\right]$$

$$= -(\tan e^z) e^z = -e^z (\tan e^z)$$

Question 6

$e^x + |e^{x^2} + \ldots\ldots + e^{x^5}$

Solution:

Let y = $e^x + |e^{x^2} + \ldots\ldots + e^{x^5}$

Define the given function for 5 terms,

Let us say, y = $e^x + e^{x^2} + e^{x^3} + e^{z^4} + e^{x^5}$

Differentiate the function with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx} e^x + \frac{d}{dx} e^{x^2} + \frac{d}{dx} e^{x^3} + \frac{d}{dx} e^{x^4} + \frac{d}{dx} e^{x^5}$$

$$= e^x + e^{x^2} \frac{d}{dx} x^2 + e^{x^3} \frac{d}{dx} x^3 + e^{x^4} \frac{d}{dx} x^4 e^{x^5} \frac{d}{dx} x^5$$

$$= e^x + e^{x^2}.2x + e^{x^3}.3x^2 + e^{x^4}.4x^3 + e^{x^5}.5x^4$$

$$= e^x + 2xe^{x^2} + 3x^2 e^{x^3} + 4x^3 e^{x^4} + 5x^4 e^{x^5}$$

Question 7

$\sqrt{e^{\sqrt{x}}}, x > 0$

Solution:

Let $y = \sqrt{e^{\sqrt{x}}}$

Or y = $\left(e^{\sqrt{x}}\right)^{\frac{1}{2}}$

Differentiate the function with respect to x we get,

$$\frac{dy}{dx} = \frac{1}{2}\left(e^{\sqrt{x}}\right)^{\frac{-1}{2}} \frac{d}{dx} e^{\sqrt{x}}$$

$$\left[\because \frac{d}{dx}\left(f(x)\right)^{n} = n\left(f(x)\right)^{n-1} \frac{d}{dx} f(x)\right]$$

$$= \frac{1}{2\sqrt{e^{\sqrt{x}}}} e^{\sqrt{x}} \frac{d}{dx}\sqrt{x}$$

$$= \frac{1}{2\sqrt{e^{\sqrt{x}}}} e^{\sqrt{x}} \frac{1}{2\sqrt{x}}$$

$$= \frac{e^{\sqrt{x}}}{4\sqrt{x}\sqrt{e^{\sqrt{x}}}}$$

Question 8

$\log(\log x), x > 1$

Solution:

Let y = $\log(\log x)$

Differentiate the function with respect to x we get,

$$\frac{dy}{dx} = \frac{1}{\log x} \frac{d}{dx}(\log x)$$

$$= \frac{1}{\log x} \cdot \frac{1}{x} = \frac{1}{x \log x}$$

Question 9

$\frac{\cos x}{\log x}, x > 0$

Solution:

Let y = $\frac{\cos x}{\log x}$

Differentiate the function with respect to x we get,

$\frac{dy}{dx} = \frac{\log x \frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(\log x)}{(\log x)^2}$ [By quotient rule]

$= \frac{\log x\,(-\sin x) - \cos x\,\frac{1}{x}}{(\log x)^2}$

$= \frac{-\left(\frac{\sin x \log x + \frac{\cos x}{x}}{(\log x)^2}\right)}{(\log x)^2}$

$= \frac{-\,(x\,sinx \log x + \cos x)}{x\,(\log x)^2}$

Question 10

***cos* (log *x* + *e*^*z*), *x* > 0**

Solution:

Let y $cos\,(\log x + e^z)$
Differentiate the function with respect to x we get,
$\frac{dy}{dx} = -\sin(\log x + e^x)\frac{d}{dx}(\log x + e^x)$
$= -\sin(\log x + e^x)\,.\left(\frac{1}{x} + e^x\right)$
$= \left(\frac{1}{x} + e^x\right)\sin(\log x + e^x)$

Exercise 5.5

Differentiate the functions with respect to x in Exercise 1 to 6

Question 1

cos x cos* 2*x cos* 3 *x

Solution:

Let y $= cos\,x\,cos\,2x\,cos\,3\,x$
Taking logs on both sides, we get
Log y = log($cos\,x\,cos\,2x\,cos\,3\,x$)
$= \log\cos x + \log\cos 2\,x + \log\cos 3x$
Now,

$$\frac{d}{dx}\log y = \frac{d}{dx}\log\cos x + \frac{d}{dx}\log\cos 2x + \frac{d}{dx}\log\cos 3x$$
$$\frac{1}{y}.\frac{dy}{dx} = \frac{1}{\cos x}\frac{d}{dx}\cos x + \frac{1}{cos2x}\ \frac{d}{dx}\cos 2x + \frac{1}{\cos 3x}\frac{d}{dx}\cos 3x$$
$$\frac{1}{y}.\frac{dy}{dx} = \frac{1}{\cos x}\ (-\sin x) + \frac{1}{cos2x}\ (-\sin 2x)\ \frac{d}{dx}2x + \frac{1}{\cos 3x}(-\sin 3x)\frac{d}{dx}\ 3x$$
$$\frac{1}{y}.\frac{dy}{dx} = \tan x\ (\tan 2x\ )\ 2 - \tan 3x\ (3)$$
$$\frac{dy}{dx} = -y\ (\tan x + 2\tan 2x + 3\tan 3x)$$
$$\frac{dy}{dx} = -cos\ x\ cos\ 2x\ cos\ 3\ x(\tan x + 2\tan 2x + 3\tan 3x)$$

Question 2

$$\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

Solution:

Let $y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

$$= \left(\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}\right)^{\frac{1}{2}}$$
$$\log y = \frac{1}{2}\ [\log(x-1) + \log(x-2) + \log(x-3) - \log(x-4) - \log(x-5)]$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x-1}\frac{d}{dx}\ (x-1) + \frac{1}{x-2}\frac{d}{dx}\ (x-2) - \frac{1}{x-3}\frac{d}{dx}\ (x-3) - \frac{1}{x-4}\frac{d}{dx}\ (x-4) - \frac{1}{x-5}\frac{d}{dx}\ (x-5)\right]$$
$$\frac{dy}{dx} = \frac{1}{2}\ y\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right]$$
$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right]$$

Question 3

$(\log\ )^{cos\ x}$

Solution:

Let y $= (\log)^{\cos x}$
Taking logs o both sides we get
$\log y = \log(\log x)^{\cos I} = \cos x \log(\log x)$
$\frac{d}{dx}\log y = \frac{d}{dx}[\cos x \log(\log x)]$
$\frac{1}{y}\frac{dy}{dx} = \cos x \frac{d}{dx}\log(\log x) + \log(\log x)\frac{d}{dx}\cos x$ [By product rule]
$\frac{1}{y}\frac{dy}{dx} = \cos x \frac{1}{\log x}\frac{d}{dx}(\log x) + \log(\log x)\,(-\sin x)$
$\frac{1}{y}\frac{dy}{dx} = \frac{\cos x}{\log x}.\frac{1}{\log x} - \sin x \log(\log x)$
$\frac{dy}{dx} = y\left[\frac{\cos x}{\log x} - \sin x \log(\log x)\right]$
$= (\log)^{\cos x}\left[\frac{\cos x}{\log x} - \sin x \log(\log x)\right]$

Question 4

$x^x - 2^{\sin x}$

Solution:

Let y = $x^x - 2^{\sin x}$
Put $u = x^x$ *and* $v = 2^{\sin x}$
$y = u - v$
$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$………… (1)
Now, $u = x^x$
$\log u = \log x^x = x \log x$
$\frac{d}{dx}\, log\, u\, \frac{d}{dx}(x \log x)$
$\frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}\log x + \log x \frac{d}{dx}x$
$\frac{1}{u}\frac{du}{dx} = 1 + \log x$
$\frac{du}{dx} = u\,(1 + \log x)$
$\frac{du}{dx} = x^x\,(1 + \log x)$ ………. (2)
Again, $v = 2^{\sin x}$
$\frac{dv}{dx} = \frac{d}{dx}2^{\sin x}$
$\frac{dv}{dx} = 2^{\sin x}\log 2\frac{d}{dx}\sin x\left[\because \frac{d}{dx}a^{f(x)} = a^{f(z)}\log a\frac{d}{dx}f(x)\right]$
$\frac{dv}{dx} = 2^{\sin x}(\log 2).\cos x = \cos x 2^{\sin x}\log 2$ ………….. (3)

Put the values from (2) and (3) in (1)

$\frac{dv}{dx} = x^x (1 + logx) - \cos x. 2^{\sin x} \log 2$

Question5

$(x+3)^2 (x+4)^3 (x+5)^4$

Solution:

let y $= (x+3)^2 (x+4)^3 (x+5)^4$

Taking logs on both sides, we get

$\log y = 2\log(x+3) + 3\log(x+4) + 4\log(x+5)^4$

Now

$\frac{d}{dx}\log y = 2\frac{d}{dx}\log (x+3) + 3\frac{d}{dx}\log (x+4) + 4\frac{d}{dx}\log (x+5)$

$\frac{1}{y}\frac{dy}{dx} = 2\frac{1}{x+3}\frac{d}{dx}(x+3) + 3\frac{1}{x+4}\frac{d}{dx}(x+4) + 4\frac{1}{x+5}\frac{d}{dx}(x+5)$

$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x+3} + \frac{3}{x+4}\frac{4}{x+5}$

$\frac{dy}{dx} = y\left(\frac{2}{x+3} + \frac{3}{x+4}\frac{4}{x+5}\right)$

$\frac{dy}{dx} = (x+3)^2 (x+4)^3 (x+5)^4 \left(\frac{2}{x+3} + \frac{3}{x+4}\frac{4}{x+5}\right)$

(Using value of y)

Differentiate the functions with respect to x in Exercise 6 to 11.

Question6

$\left(x+\frac{1}{x}\right)^x + x^{\left(x+\frac{1}{x}\right)}$

Solution:

Let y $= \left(x+\frac{1}{x}\right)^x + x^{\left(x+\frac{1}{x}\right)}$

Put $\left(x+\frac{1}{x}\right)^x = u$ and $x^{\left(x+\frac{1}{x}\right)} = v$

$y = u + v$

$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$............ (1)

Now $u = \left(x+\frac{1}{x}\right)^x$

$\log u = \log \left(x + \frac{1}{x}\right)^x = x \log\left(x + \frac{1}{x}\right)$

$\frac{1}{u}\frac{du}{dx} = x.\frac{1}{\left(x+\frac{1}{x}\right)}\frac{d}{dx}\left(x + \frac{1}{x}\right) + \log\left(x + \frac{1}{x}\right).1$

$\frac{1}{u}\frac{du}{dx} = x.\frac{1}{\left(x+\frac{1}{x}\right)}\left(x + \frac{1}{x^2}\right) + \log\left(x + \frac{1}{x}\right).1$

$\frac{du}{dx} = u\left[\frac{x^2-1}{x^2+1} + \log\left(x + \frac{1}{x}\right)\right]$

$= \left(x + \frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1} + \log\left(x + \frac{1}{x}\right)\right]$ .... (2)

Again $v = x^{\left(x+\frac{1}{x}\right)}$

$\log v = \log \boldsymbol{x}^{\left(x+\frac{1}{x}\right)} = \left(x + \frac{1}{x}\right)\log x$

$\frac{1}{v}\frac{dv}{dx} = \left(x + \frac{1}{x}\right)\frac{\mathbf{1}}{\boldsymbol{x}} + \log x\left(\frac{-1}{x^2}\right)$

$\frac{dv}{dx} = v\left[\frac{1}{x}\left(x + \frac{1}{x}\right) - \frac{1}{x^2}\log x\right]$

$\frac{dv}{dx} = x^{\left(x+\frac{1}{x}\right)}\left[\frac{1}{x}\left(x + \frac{1}{x}\right) - \frac{1}{x^2}\log x\right]$ .... (3)

Put the values from (2) and (3) in (1),

$\frac{dy}{dx} = \left(x + \frac{1}{x}\right)^z\left[\frac{x^2-1}{x^2+1} + \log\left(x + \frac{1}{x}\right)\right] + x^{\left(x+\frac{1}{x}\right)}\left[\frac{1}{x}\left(x + \frac{1}{x}\right) - \frac{1}{x^2}\log x\right]$

Question7

$(\boldsymbol{log\ x})^{\boldsymbol{x}} + \boldsymbol{x}^{\boldsymbol{log\ x}}$

Solution:

Let $= (\log x)^x + x^{log\ x} = u + v\ where\ u = (\log x)^x\ and\ v = x^{\log x}$

$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$............ (1)

Now $u = (log\ x)^x$

$\log u = log\ (log\ x)^x = x\log(\log x)$

$\frac{d}{dx}\log y = \frac{d}{dx}[x\log(\log x)]$

$\frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}[\log(\log x)] + \log(\log x)\frac{d}{dx}x$

$\frac{1}{u}\frac{du}{dx} = x\frac{1}{logx}\frac{d}{dx}\log x + \log(\log x).1$

$\frac{1}{u}\frac{du}{dx} = x\frac{1}{logx}\frac{1}{x} + \log(\log x)$

$$\frac{du}{dx} = u\left[\frac{1}{\log x} + \log(\log x)\right]$$

$\frac{du}{dx} = (\log x)^x \left[\frac{1}{\log x} + \log(\log x)\right]$ .... (2)

Again $v = x^{\log x}$

$\log v = \log x^{\log x} = \log x \,.\log x = (\log x)^2$

$$\frac{d}{dx}\log v = \frac{d}{dx}(\log x)^2$$

$$\frac{1}{v}\frac{dv}{dx} = 2\log x\,\frac{d}{dx}(\log x)$$

$$\frac{1}{v}\frac{dv}{dx} = 2\log x\,\frac{1}{x}$$

$$\frac{dv}{dx} = v\left(\frac{2}{x}\log x\right) x^{\log x}\,\frac{2}{x}\,log\,x$$

$\frac{dv}{dx} = 2x^{log\,x}\log x$ .... (3)

Put the values from (2) and (3) in (1)

$$\frac{dy}{dx} = (\log x)^x\left[\frac{1}{\log x} + \log(\log x)\right] + 2\,x^{\log x - 1}\log x$$

$$\frac{dy}{dx} = (\log x)^x\left[\frac{1 + \log x\log(\log x)}{\log x}\right] + 2\,x^{\log x - 1}\log x$$

Question8

$(\sin x)^x + \sin^{-1}\sqrt{x}$

Solution:

Let $y = (\sin x)^x + \sin^{-1}\sqrt{x} = u + v$ where $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$

$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$........... (1)

Now $u = (\sin x)^x$

$\log u = \log(\sin x)^x = \log(\sin x)$

$$\frac{d}{dx}\log u = \frac{d}{dx}[x\log(\sin x)]$$

$$\frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}[\log(\sin x)] + \log(\sin x)\frac{d}{dx}x$$

$$\frac{1}{u}\frac{du}{dx} = x\frac{1}{\sin x}\frac{d}{dx}\sin x + \log(\sin x).1$$

$$\frac{1}{u}\frac{du}{dx} = x\frac{1}{\sin x}\cos x + \log(\sin) = \mathrm{x}\cot \mathrm{x} + \log\sin \mathrm{x}$$

$$\frac{du}{dx} = u\,[x\cot x + \log\sin x]$$

$\frac{du}{dx} = (\sin x)^x\,[x\cot x + \log\sin x]$............ (2)

Again $v = \ sin^{-1}\sqrt{x}$
$\log v = \log sin^{-1}\sqrt{x}$
$\frac{dv}{dx} = \frac{1}{\sqrt{1\,(\sqrt{x})^2}}\frac{d}{dx}\sqrt{x}\left[\because \frac{d}{dx}\ sin^{-1} f(x) = \frac{1}{\sqrt{1-\left(f\,(x)\right)^2}}\frac{d}{dx}\ f\,(x)\right]$
$\frac{dv}{dx} = \frac{1}{\sqrt{1-x}}.\frac{1}{2\sqrt{x}}$
$= \frac{1}{2\sqrt{x}\,\sqrt{1-x}}$
$= \frac{1}{2\sqrt{x-x^2}}$ .... (3)
Put the values from (2) and (3) in (1)
$\frac{dy}{dx} = (\sin x)^x\ [x\cot x + \log\sin x] + \frac{1}{2\sqrt{x-x^2}}$

Question9

$\boldsymbol{x^{\sin x} + \ (\sin x)^{cos\,x}}$

Solution:

Let $y\ x^{\sin\,x} + \ (\sin x)^{cos\,x}$
Put $u = \ x^{\sin\,x}$ and $v = \ (\sin x)^x$, we get $y = u + v$
$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$............ (1)
Now $u = x^{\sin\,x}$
$\log u = \log x^{\sin\,x}\ \sin x\log x$
$\frac{d}{dx}\log u = \frac{d}{dx}\ (\sin x\log x)$
$\frac{1}{u}\frac{du}{dx} = \sin x\ \frac{d}{dx}\log x + \log x\ \frac{d}{dx}\sin x$
$\frac{1}{u}\frac{du}{dx} = \sin x\ \frac{1}{x} + \log x\ (\cos x)$
$\frac{du}{dx} = u\ \left(\frac{sin\ x}{x} + \cos x\log x\right)$
$\frac{du}{dx} = \ x^{\sin\,x}\ \left(\frac{sin\ x}{x} + \cos x\log x\right)$ .... (2)
Again $v = \ (\sin x)^{cos\,x}$
$\log v = \log(\sin x)^{cos\,x} = \cos x\log\sin x$
$\frac{d}{dx}\log v = \frac{d}{dx}\ [\cos x\log\ (\sin x)]$
$\frac{1}{v}\frac{dv}{dx} = \cos x\ \frac{d}{dx}\log\sin x + \log\sin x\ \frac{d}{dx}\cos x$

$$\frac{1}{v}\frac{dv}{dx} = \cos x\ \frac{1}{\sin x}\ \frac{d}{dx}\sin x + \log \sin x\ (-\sin x)$$
$$\frac{1}{v}\frac{dv}{dx} = \cot x \cos x - \sin x \log \sin x$$
$$\frac{dv}{dx} = v\ (\cot x \cos x - \sin x \log \sin x)$$
$$\frac{dv}{dx} = \ (\sin x)^{\cos x}(\cot x \cos x - \sin x \log \sin x)\ (\text{Using value of v}) \ldots\ldots\ldots. (3)$$

Put values from (2) and (3) in (1)

$$\frac{dv}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \cos x \log x\right) + \ (\sin x)^{\cos x}\ (\cot x . \cos x - \sin x \log \sin x)$$

Question10

$$x^{x\cos x} + \frac{x^2+1}{x^2-1}$$

Solution:

Let $x^{x\cos x} + \dfrac{x^2+1}{x^2-1}$

Put $u = x^{x\cos x}$ and $v = \frac{x^2+1}{x^2-1}$ we get $y = u + v$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \ldots\ldots\ldots\ldots (1)$$

Now $u = x^{x\cos x}$

$$\log u = \log x^{x\cos x} = x \cos x \log x$$
$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}\ (x \cos x \log x)$$
$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}\ (x).\cos x + x\ \frac{d}{dx}\ (\cos x)\log x + x \cos x\ \frac{d}{dx}\ (\log x)$$
$$\frac{1}{u}\frac{du}{dx} = 1.\cos x \log x + x\ (-\sin x)\log x + x\cos x\ \frac{1}{x}$$
$$\frac{du}{dx} = u\ (\cos x \log x - x \sin x \log x + \cos x)$$
$$\frac{du}{dx} = x^{x\cos x}(\cos x \log x - x \sin x \log x + \cos x)\ \ldots\ldots\ldots (2)$$

Again $v = \frac{x^2+1}{x^2-1}$

$$\frac{dv}{dx} = \frac{(x^2-1)\ \frac{d}{dx}\ (x^2+1) - (x^2+1)\ \frac{d}{dx}\ (x^2-1)}{(x^2-1)^2}$$
$$\frac{dv}{dx} = \frac{(x^2-1)2x - \ (x^2+1)\ 2x}{(x^2-1)^2}$$
$$\frac{dv}{dx} = \frac{2x^3 - \ 2x - 2x^3 - 2x}{(x^2-1)^2}$$

$\frac{dv}{dx} = \frac{-4x}{(x^2-1)^2}$ ………… (3)

Put the value from (2) and (3) in (1)

$\frac{dv}{dx} = x^{x\cos x}\left(\cos x \log x - x \sin x \log x + \cos x\right) + \frac{-4x}{(x^2-1)^2}$

Question11

$(x\cos x)^x + (x\sin x)^{\frac{1}{z}}$

Solution:

Let $y = (x\cos x)^x + (x\sin x)^{\frac{1}{z}}$

Put $u = (x\cos x)^x$ and $v = (x\sin x)^{\frac{1}{z}}$ we get $y = u + v$

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$………… (1)

Now $u = (x\cos x)^x$

$\log u = \log(x\cos x)^x = x\log(x\cos x)$

$\log u = x(\log x + \log\cos x)$

$\frac{d}{dx}\log u = \frac{d}{dx}\{x(\log x + \log\cos x)\}$

$\frac{1}{u}\frac{du}{dx} = \left[\frac{1}{x} + \frac{1}{\cos x}.(-\sin x)\right] + (\log x + \log\cos x).1$

$\frac{du}{dx} = u\left[1 - x\tan x + \log(x\cos x)\right]$

$\frac{du}{dx} = (x\cos x)^x\left[1 - x\tan x + \log(x\cos x)\right]$ ………. (2)

Again $v = (x\sin x)^{\frac{1}{z}}$

$\log v = \log(x\sin x)^{\frac{1}{z}} = \frac{1}{x}\log(x\sin x)$

$\log v = \frac{1}{x}(\log x + \log\sin x)$

$\frac{d}{dx}\log v = \frac{d}{dx}\left\{\frac{1}{x}(\log x + \log\sin x)\right\}$

$\frac{1}{v}\frac{dv}{dx} = \frac{1}{x}\left[\frac{1}{x} + \frac{1}{\sin x}.\cos x\right] + (\log x + \log\sin x)\left(\frac{-1}{x^2}\right)$

$\frac{dv}{dx} = v\left[\frac{1}{x^2} + \frac{\cot x}{x} - \frac{\log(x\sin x)}{x^2}\right]$

$\frac{dv}{dx} = (x\sin x)^{\frac{1}{z}}\left[\frac{1}{x^2} + \frac{\cot x}{x} - \frac{\log(x\sin x)}{x^2}\right]$……… (3)

Put the value form (2) and (3) in (1)

$\frac{dv}{dx} = (x\cos x)^x\left[1 - x\tan x + \log(x\cos x)\right] + (x\sin x)^{\frac{1}{z}}\left[\frac{1}{x^2} + \frac{\cot x}{x} - \frac{\log(x\sin x)}{x^2}\right]$

Find $\frac{dy}{dx}$ in the following Exercise 12 to 15

Question12

$x^y + y^x = 1$

Solution:

Given : $x^y + y^x = 1$

$u + v = 1$ where $u = x^y$ and $v = y^x$

$\frac{d}{dx} u + \frac{d}{dx} v = \frac{d}{dx} 1$

$\frac{du}{dx} + \frac{dv}{dx} = 0$ .... (1)

Now $u = x^y$

$\log u = \log x^y = y \log x$

$\frac{d}{dx} \log u = \frac{d}{dx} (y \log x)$

$\frac{1}{u}\frac{du}{dx} = y\frac{1}{x}\log x + \log x \frac{dy}{dx}$

$\frac{du}{dx} = u\left(\frac{y}{x} + \log x \frac{dy}{dx}\right)$

$\frac{du}{dx} = x^y\left(\frac{y}{x} + \log x \frac{dy}{dx}\right) = x^y \frac{y}{x} + x^y \log x . \frac{dy}{dx}$

$\frac{du}{dx} = x^{y-1} y + x^y \log x . \frac{dy}{dx}$ .... (2)

Again $v = y^x$

$\log v = \log y^x = x \log y$

$\frac{d}{dx} \log v = \frac{d}{dx} (x \log y)$

$\frac{1}{v}\frac{d}{dx} = x\frac{d}{dx}\log y + \log y \frac{d}{dx} x$

$\frac{1}{v}\frac{d}{dx} = x . \frac{1}{y}\frac{dy}{dx} + \log y . 1$

$\frac{dv}{dx} = v\left(\frac{x}{y}\frac{dy}{dx} + \log y\right)$

$\frac{dv}{dx} = y^x\left(\frac{x}{y}\frac{dy}{dx} + \log y\right) = y^x \frac{x}{y}\frac{dy}{dx} + y^x \log y$

$\frac{dv}{dx} = y^{x-1} x \frac{dy}{dx} + y^x \log y$ .... (3)

Put values form (2) and (3) in (1),

$x^{y-1} y + x^y \log x \frac{dy}{dx} + y^{x-1} x \frac{dy}{dx} + y^x \log y = 0$

$\frac{dy}{dx}(x^y \log x + y^{x-1} x) = -x^{y-1} y - y^x \log y$

$$\frac{dy}{dx} = \frac{-(x^{y-1}\, y - y^{x}\log y)}{x^{y}\log x + \; y^{x-1}\, x}$$

Question13

$y^z = x^y$

Solution:

Given $y^z = x^y$
$x^y = y^x$
$\log x^y \log = y^x$
$y\log x = x\log y$
$$\frac{d}{dx}(y\log x) = \frac{d}{dx}(x\log y)$$
$$y.\frac{1}{x} + \log x.\frac{dy}{dx} = x.\frac{1}{y}\frac{dy}{dx} + \log y.1$$
$$\left(\log x - \frac{x}{y}\right)\frac{dy}{dx} = \log y - \frac{y}{x}$$
$$\left(\frac{y\log x - x}{y}\right)\frac{dy}{dx} = \frac{x\log y - y}{x}$$
$$\frac{dy}{dx} = \frac{y\,(x\log y - y)}{x\,(y\log x - x)}$$

Question14

$(\cos x)^y = (\cos y)^x$

Solution:

Given $(\cos x)^y = (\cos y)^x$
$\log(\cos x)^y = \log(\cos y)^x$
$y\log\cos x = x\log\cos y$
$$\frac{d}{dx}(y\log\cos x) = \frac{d}{dx}(x\log\cos y)$$
$$y\frac{d}{dx}\log\cos x + \log\cos x\frac{dy}{dx} = x\frac{d}{dx}\log\cos y + \log\cos y\frac{d}{dx}x$$
$$y\frac{1}{\cos x}\frac{d}{dx}\cos x + \log\cos x\frac{dy}{dx} = x\frac{1}{\cos y}\frac{d}{dx}\cos y + \log\cos y$$
$$y\frac{1}{\cos x}(-\sin x) + \log\cos x\frac{dy}{dx} = x\frac{1}{\cos y}\left(-\sin y\frac{dy}{dx}\right) + \log\cos y$$
$$-y\tan x + \log\cos x\frac{dy}{dx} = -x\tan y.\frac{dy}{dx} + \log\cos y$$

$x \tan y \frac{dy}{dx} + \log \cos x \,.\frac{dy}{dx} = y \tan x \log \cos y$

$\frac{dy}{dx}(x \tan y + \log \cos x) = y \tan x + \log \cos y$

$\frac{dy}{dx} = \frac{y \tan x + \log \cos y}{x \tan y + \log \cos x}$

Question15

$\boldsymbol{xy = e^{x-y}}$

Solution:

Given $xy = e^{x-y}$

$\log xy = \log e^{x-y}$

$\log x + \log y = (x - y)\log e$

$\log x + \log y = (x - y)\ [\because \log e = 1]$

$\frac{d}{dx}\log x + \frac{d}{dx}\log y = \frac{d}{dx}(x - y)$

$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 1 - \frac{dy}{dx}$

$\frac{1}{y}.\frac{dy}{dx} + \frac{dy}{dx} = 1 - \frac{1}{x}$

$\frac{dy}{dx}\left(\frac{1}{y} + 1\right) = \frac{x-1}{x}$

$\frac{dy}{dx}\left(\frac{1+y}{y}\right) = \frac{x-1}{x}$

$\frac{dy}{dx} = \frac{y(x-1)}{x(1+y)}$

Question16

Find the derivative of the function given by $f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$ hence $f(1)$

Solution:

Given $f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$ ……….. (1)

$\log f(x) = \log\ (1+x)(1+x^2)(1+x^4)(1+x^8)$

$\frac{1}{f(x)}\frac{d}{dx}f(x) = \frac{1}{1+x}\frac{d}{dx}(1+x) + \frac{1}{1+x^2}\frac{d}{dx}(1+x^2) + \frac{1}{1+x^4}\frac{d}{dx}(1+x^4) + \frac{1}{1+x^8}\frac{d}{dx}$
$(1+x^8)$

$\frac{1}{f(x)}f(x) = \frac{1}{1+x}.1 + \frac{1}{1+x^2}2x + \frac{1}{1+x^4}4x^3 + \frac{1}{1+x^8}8x^7$

$f(x) - f(x)\left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right]$

Put the value of (x) from (1)

$f(x) - (1+x)(1+x^2)(1+x^4)(1+x^8)\left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right]$

Now, find for f (1):

$f(x) - (1+x)(1+x^2)(1+x^4)(1+x^8)\left[\frac{1}{1+1} + \frac{2\times 1}{1+1^2} + \frac{4\times 1^3}{1+1^4} + \frac{8\times 1^7}{1+1^8}\right]$

$f(1) - (2)(2)(2)(2)\left[\frac{1}{2} + \frac{2}{2} + \frac{4}{2} + \frac{8}{2}\right]$

$f(1) = 16\left[\frac{15}{2}\right]$

$= 8 \times 15$

$= 120$

Question17

Differentiate $(x^2 - 5x + 8)(x^3 + 7x + 9)$ in three ways mentioned below:
(i) by using product rule.
(ii) by expanding the product to obtain a single polynomial
(iii) by logarithmic differentiation.
Do they all give the same answer?

Solution:

Let $(x^2 - 5x + 8)(x^3 + 7x + 9)$

(i) Using product rule:

$\frac{dy}{dx} = (x^2 - 5x + 8)\frac{d}{dx}(x^3 + 7x + 9) + (x^3 + 7x + 9)\frac{d}{dx}(x^2 - 5x + 8)$

$\frac{dy}{dx} = (x^2 - 5x + 8)(3x^2 + 7) + (x^3 + 7x + 9)(2x - 5)$

$\frac{dy}{dx} = 3x^4 + 7x^2 - 15x^3 - 35x + 24x^2 + 56 + 2x^4 - 5x^3 + 14x^2 - 35x + 18x - 45$

$\frac{dy}{dx} = 5x^4 - 20x^3 + 45x^2 + 11$

(ii) Explain the product to obtain a single polynomial

$y = (x^2 - 5x + 8)(x^3 + 7x + 9)$

$y = x^5 + 7x^3 + 9x^2 - 5x^4 - 35x^2 - 45x + 8x^3 + 56x + 72$

$y = x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72$

$\frac{dy}{dx} = 5x^4 - 20x^3 + 45x^2 - 52x + 11$

(iii) Logarithmic differentiation

$y = (x^2 - 5x + 8)(x^3 + 7x + 9)$

$\log y - \log(x^2 - 5x + 8) + \log(x^3 + 7x + 9)$

$\frac{d}{dx}\log y = \frac{d}{dx}\log(x^2 - 5x + 8) + \frac{d}{dx}\log(x^3 + 7x + 9)$

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2 - 5x + 8}\frac{d}{dx}(x^2 - 5x + 8)\frac{1}{x^3 + 7x + 9}\frac{d}{dx}(x^3 + 7x + 9)$

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2 - 5x + 8}(2x - 5)\frac{1}{x^3 + 7x + 9}(3x^2 + 7)$

$\frac{dy}{dx} - y\left[\frac{2x - 5}{x^2 - 5x + 8} + \frac{3x^2 + 7}{x^3 + 7x + 9}\right]$

$\frac{dy}{dx} - y\left[\frac{2x - 5}{x^2 - 5x + 8} + \frac{3x^2 + 7}{x^3 + 7x + 9}\right]$

$\frac{dy}{dx} - y\left[\frac{(2x - 5)(x^3 + 7x + 9)(3x^2 + 7)(x^2 - 5x + 8)}{(x^2 - 5x + 8)(x^3 + 7x + 9)}\right]$

$\frac{dy}{dx} = y\left[\frac{2x^4 + 14x^2 + 18x - 5x^3 - 35x - 45 + 3x^4 - 15x^3 + 24x^2 + 7x^2 - 35x + 56}{(x^2 - 5x + 8)(x^3 + 7x + 9)}\right]$

$\frac{dy}{dx} = y\left[\frac{5x^4 - 20x^3 + 45x^2 - 52x + 11}{(x^2 - 5x + 8)(x^3 + 7x + 9)}\right]$

$\frac{dy}{dx} = (x^2 - 5x + 8)(x^3 + 7x + 9)\left[\frac{5x^4 - 20x^3 + 45x^2 - 52x + 55}{(x^2 - 5x + 8)(x^3 + 7x + 9)}\right]$ [Using value of y]

$\frac{dy}{dx} = 5x^4 - 20x^3 + 45x^2 - 52x + 11$

Therefore the value of dy/dx is same obtained by three different methods.

Question18

If u, v and w are functions of x, then show that
$\frac{d}{dx}(u.v.w.) = \frac{du}{dx}v.w. + u\frac{dv}{dx}w + u.v\frac{dw}{dx}$
in two ways – first by repeated application of product rule, second by logarithmic differentiation.

Solution:

Give u,v and w are functions of x.

To prove: $\frac{d}{dx}(u.v.w.) = \frac{du}{dx}v.w. + u\frac{dv}{dx}w + u.v\frac{dw}{dx}$

Way 1: By repeated application of product rule

L.H.S.

$\frac{d}{dx}(u.v.w.) = \frac{d}{dx}[(uv).w]$

$= uv.\frac{d}{dx}w + w\frac{d}{dx}(uv)$

$= uv.\frac{d}{dx}w\left[u\frac{d}{dx}v + v\frac{d}{dx}u\right]$

$= uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$

$= \frac{du}{dx}.v.w + u.\frac{dv}{dx}.w + u.v.\frac{dw}{dx}$

= R.H.S.

Hence proved.

Way 2: By logarithmic differentiation

Let $y = uvw$

$\log y = \log(u.v.w.)$

$\log y - \log u + \log v + \log w$

$\frac{d}{dx}\log y = \frac{d}{dx}\log u + \frac{d}{dx}\log v + \frac{d}{dx}\log w$

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}$

$\frac{dy}{dx} = y\left[\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right]$

Put y = uvw, we get

$\frac{d}{dx}(u.v.w) = uvw\left[\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right]$

$\frac{d}{dx}(u.v.w)\frac{du}{dx}v.w + u\frac{dv}{dx}w + u.v.\frac{dw}{dx}$

Hence proved

Exercise 5.6

If x and y are connected parametrically by the equation gives in Exercise 1 to 10 eliminating the parameter the parameter, find dy/dx

Question 1

$\boldsymbol{x = 2at^2. y = at^4}$

Solution:

Given function are $x = 2at^2$ and $y = at^4$

$\frac{dx}{dt} = \frac{d}{dx}(2at^2)$

$\frac{dx}{dt} = 2a\frac{d}{dt}(t^2)$

$= 2a .2t\ 4at$ and

$\frac{dx}{dt} = \frac{d}{dx}(at^4)$

$\frac{dy}{dt} = a\frac{d}{dt}(at^4) = a\,4t^3 = 4at^3$

Now

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4\ at^3}{4at} = t^2$$

Question 2

$$\boldsymbol{x = a\ cos\ \theta . y = b\ cos\ \theta}$$

Solution:

Given functions are $x = a\ cos\ \theta$ and $y = b\ cos\ \theta$

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(a\cos\theta)$$

$$\frac{dx}{d\theta} = a\frac{d}{d\theta}(a\cos\theta)$$

$$\frac{dx}{d\theta} = -a\sin\theta$$

And

$$\frac{dy}{d\theta} = a\frac{d}{d\theta}(b\cos\theta)$$

$$\frac{dy}{d\theta} = b\frac{d}{d\theta}(\cos\theta)$$

$$\frac{dy}{d\theta} = -b\sin\theta$$

Now

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-a\ sin\theta}{-b\ sin\theta} = \frac{b}{a}$$

Question 3

$$\boldsymbol{x = sin\ t, y = cos\ 2t}$$

Solution:

Given functions $x = sin\ t$ and $y = cos\ 2t$

$\frac{dx}{dt} = \cos t$ And

$$\frac{dx}{dt} - \sin 2t\frac{d}{dt}(2t) = -2\sin 2t$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-2\sin 2\ t}{\cos t} = \frac{-2 \times 2\sin t\cos t}{\cos t} = -4\sin t$$

Question 4

$$x = 4t, y = \frac{4}{t}$$

Solution:

Given functions $x = 4t$ and $y = \frac{4}{t}$

$\frac{dx}{dt} = \frac{d}{dt}(4t) = 4\frac{d}{dt}t = 4$

And

$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{4}{t}\right)$

$= \frac{t\frac{d}{dt}4 - 4\frac{d}{dt}t}{t^2}$

$\Rightarrow \frac{dy}{dt} = \frac{t \times 0 - 4 \times 1}{t^2} = \frac{4}{t^2}$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\frac{4}{t^2}}{4} = \frac{-1}{t^2}$

Question 5

$$x = \cos\theta - \cos 2\theta, y = \sin\theta - \sin 2\theta$$

Solution:

Given functions $x = \cos\theta - \cos 2\theta$ and $y = \sin\theta - \sin 2\theta$

$\frac{dx}{dt} = \frac{d}{dt}\cos\theta - \frac{d}{d\theta}\cos 2\theta$

$\frac{dx}{d\theta} = -\sin\theta - (-\sin 2\theta)\frac{d}{d\theta}2\theta$

$\frac{dx}{d\theta} = -\sin\theta - (-\sin 2\theta)\,2$

$\frac{dx}{d\theta} = 2\sin 2\theta - \sin\theta$

And

$\frac{dy}{d\theta} = \frac{d}{d\theta}\sin\theta - \frac{d}{d\theta}\sin 2\theta$

$\frac{dy}{d\theta} = \cos\theta - \cos 2\frac{d}{d\theta}2\theta$

$\frac{dy}{d\theta} = \cos\theta - \cos 2\theta \times 2$

$\frac{dx}{d\theta} = \cos\theta - 2\cos 2\theta$

Now $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta - 2\cos 2\theta}{2\sin 2\theta - \sin\theta}$

Question 6

$x = a\,(\theta - \sin\theta), y = a\,(1 + \cos\theta)$

Solution:

Given functions are $x = a\,(\theta - \sin\theta)$ and $y = a\,(1 + \cos\theta)$

$\frac{dx}{d\theta} = a\frac{d}{d\theta}\,(\theta - \sin\theta)$

$\frac{dx}{d\theta} = a\left[\frac{d}{d\theta}\,\theta - \frac{d}{d\theta}\sin\theta\right]$

$\frac{dx}{d\theta} = a\,(1 - \cos\theta)$

$\frac{dy}{d\theta} = a\frac{d}{d\theta}\,(1 + \cos\theta)$

$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}\,(1) + \frac{d}{d\theta}\,cos\theta\right]$

$\frac{dy}{d\theta} = a\,(0 - \sin\theta)$

$= -a\sin\theta$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-a\sin\theta}{a\,(1 - \cos\theta)} = \frac{-\sin\theta}{1 - \cos\theta}$

$= -\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\,sin^2\frac{\theta}{2}}$

$= -\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$

$= -cot\,\frac{\theta}{2}$

Question 7

$x = \frac{sin^3 r}{\sqrt{cos\,2\,t}}, y = \frac{cos^3\,t}{\sqrt{cos\,2\,t}}$

Solution:

Given function: $x = \frac{sin^3 r}{\sqrt{cos\,2\,t}}$ and $y = \frac{cos^3\,t}{\sqrt{cos\,2\,t}}$

$\frac{dx}{dt} = \frac{\sqrt{cos\,2t}\,\frac{d}{dt}(sin^3\,t) - sin^3\,t.\frac{d}{dt}(\sqrt{cos\,2\,t})}{(\sqrt{cos\,2t})^2}$ [By quotient rule]

$$= \frac{\sqrt{\cos 2t}\,.3\,sin^2 t\,\frac{d}{dt}(\sin t) - sin^3 t\,\frac{1}{2}(\cos\ 2r)^{\frac{-1}{2}}\frac{d}{dt}(\cos 2\,t)}{\cos 2t}$$

$$= \frac{3\,sin^2\,t\cos 2t + sin^3 t.\sin 2t}{\cos 2t}$$

$$= \frac{sin^2\,t\cos t\,(3\cos 2t + 2sin^2 t)}{(\cos 2t)^{\frac{3}{2}}}$$

And $\frac{dx}{dt} = \frac{\sqrt{cos\ 2t}\,\frac{d}{dt}(cos^3\,t) - cos^3\,t.\frac{d}{dt}(\sqrt{cos\ 2\,t})}{(\sqrt{cos\ 2t})^2}$ [By quotient rule]

$$= \frac{\sqrt{\cos 2\,t}\,.3\,cos^2\,t\,\frac{d}{dt}(\cos t) - cos^3\,t\,\frac{1}{2}(\cos\ 2r)^{\frac{-1}{2}}\frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2\,t}\,.3\,cos^2\,(-\sin t) - \frac{cos^3\,t}{2\sqrt{\cos 2t}}(-2\sin 2t)}{\cos 2t}$$

$$= \frac{-3\,sin^2\,t\cos 2t + sin^3 t.\sin 2t}{(\cos 2t)^{\frac{3}{2}}}$$

$$= \frac{-3\,cos^2\,t\,sin\,t\cos 2t + cos^3 t.\sin t\cos t}{(\cos 2t)^{\frac{3}{2}}}$$

$$= \frac{\sin t\,cos^2\,t(2\,cos^2\,t - 3\cos 2t)}{(\cos 2t)^{\frac{3}{2}}}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{\sin t\,cos^2\,t(2\,cos^2\,t - 3\cos 2t)}{(\cos 2t)^{\frac{3}{2}}}}{\frac{sin^2 t\,cost\,(3\cos 2t + 2\,sin^2\,t)}{(\cos 2t)^{\frac{3}{2}}}}$$

$$= \frac{\cos t\,[2\,cos^2\,t - 3\,(2\,cos^2\,t - 1)]}{\sin t\,[3\,(1 - 2\,sin^2) + 2\,sin^2 t]}$$

$$= \frac{\cos t\,(3 - 4\,cos^2 t)}{\sin t\,(3 - 4\,sin^2 t)}$$

$$= \frac{-(4\,cos^2\,t - 3\cos t)}{3\sin t - 4\,sin^3\,t}$$

$$= \frac{-\cos 3\,t}{\sin 3t} = -\cot 3t$$

Question 8

$$x = a\left(cos\ t + \log\tan\frac{t}{2}\right), y = a\sin t$$

Solution:

Given functions are $x = a\left(cos\ t + \log\tan\frac{t}{2}\right)$ and $y = a\sin t$

$$\frac{dx}{dt} = a\left[-\sin t\ \frac{1}{\tan\frac{t}{2}}\frac{d}{dt}\left(\tan\frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \frac{1}{\tan\frac{t}{2}}.sec^2\frac{t}{2}.\frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}.\frac{1}{cos^2\frac{t}{2}}.\frac{1}{2}\right]$$

$$= a\left[-\sin t + .\frac{1}{2\ sin\frac{t}{2}\ cos\frac{t}{2}}\right]$$

$$= a\left[-\sin t + \frac{1}{\sin t}\right]$$

$$= a\left[\frac{1}{\sin t} - \sin t\right] = a\left(\frac{1 - sin^2\ t}{\sin t}\right) = \frac{a\ cos^2\ t}{\sin t}$$

And $\frac{dy}{dt} = a\cos t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\cos t}{\left(\frac{a\ cos^2\ t}{\sin t}\right)}$$

$$= \frac{\sin t}{\cos t} = \tan t$$

Question 9

$\boldsymbol{x = a\ sec\ \theta, y = b\ tan\ \theta}$

Solution:

Given function are $x = a\ sec\ \theta$ and $y = b\ tan\ \theta$

$\frac{dx}{d\theta} = a\sec\theta\tan\theta$ and

$$\frac{dx}{d\theta} = sec^2\ \theta$$

Now,

$$\frac{dy}{dx} = \frac{dy\ /d\theta}{dx/d\theta} = \frac{b\ sec^2\theta}{a\sec\theta\tan\theta}$$

$$= \frac{b\sec\theta}{a\tan\theta}$$

$= \dfrac{b.\frac{1}{\cos\theta}}{a\,.\frac{\sin\theta}{\cos\theta}}$

$= \frac{b}{\cos\,\theta}.\frac{\cos\,\theta}{\sin\,\theta}$

$= \dfrac{b}{a\sin\theta}$

$= \dfrac{b}{a}\ cosec\ \theta$

Question 10

$\boldsymbol{x = a\,(cos\ \theta +\ \theta\ sin\ \theta), y = a\,(sin\ \theta -\ \theta\ cos\ \theta)}$

Solution:

Given function are $x = a\,(cos\ \theta +\ \theta\ sin\ \theta)$ and $y = a\,(sin\ \theta -\ \theta\ cos\ \theta)$

$\dfrac{dx}{d\theta} = a\ (-\sin\theta +\ \theta\cos\theta + \sin\theta\ .1)$

$= a\ \theta\cos\theta$

And

$\frac{dy}{d\theta} = a\left[\cos\theta - \left\{\theta\left(-\sin\theta\right) + \cos\theta\ .1\right\}\right]$

$= a\,[\cos\theta +\ \theta\sin\theta - \cos\theta]$

$= a\ \theta\sin\theta$

$\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{a\ \theta\sin\theta}{a\ \theta\cos\theta} = \tan\theta$

Question 11

if $\boldsymbol{x =\ \sqrt{a^{\,sin-t}}.\ y =\ \sqrt{a^{cos-1t}}}$ **. Show that** $\boldsymbol{\frac{dy}{dx} = \frac{-y}{x}}$

Solution:

$x = \sqrt{a^{\ sin-1\,t}}.= \left(a^{\,sin^{-1}t}\right)^{\frac{1}{2}}$

$=\ a^{\frac{1}{2}\,sin^{-1}t}$

And

$y = \sqrt{a^{\,cos^{-1}t}} =\ \left(a^{\,cos^{-1}t}\right)^{\frac{1}{2}}$

$=\ a^{\frac{1}{2}\,cos^{-1}\ t}$

Now

$\frac{dx}{dt} = a^{\frac{1}{2}sin^{-1}t}\log a\frac{d}{dt}\left(\frac{1}{2}sin^{-1}t\right)$

$= a^{\frac{1}{2}sin^{-1}t}\log a\frac{1}{2}\frac{1}{\sqrt{1-t^2}}$

And $\frac{dy}{dt} = a^{\frac{1}{2}cos^{-1}t}\log a\frac{d}{dt}\left(\frac{1}{2}cos^{-1}t\right)$

$= a^{\frac{1}{2}cos^{-1}t}\log a\frac{1-}{2}\frac{1}{\sqrt{1-t^2}}$

Now,

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a^{\frac{1}{2}cos^{-1}t}\log a\frac{1}{2}\frac{-1}{\sqrt{1-t^2}}}{a^{\frac{1}{2}sin^{-1}t}\log a\frac{1}{2}\frac{1}{\sqrt{1-t^2}}}$

$= \frac{-a^{\frac{1}{2}cos^{-1}t}}{a^{\frac{1}{2}sin^{-1}t}} = \frac{-y}{x}$

Exercise 5.7

Find the second order derivatives of the functions given in Exercise 1 to 10

Question 1

$x^2 + 3x + 2$

Solution:

Let y $= x^2 + 3x + 2$

First derivative

$\frac{dy}{dx} = 2x + 3 \times 1 + 0 = 2x + 3$

Second derivative:

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = 2 \times 1 + 0 = 2$

Question 2

x^{20}

Solution:

Let $= y = x^{20}$

Derivate y with respect to x we get

$\frac{dy}{dx} = 20x^{19}$

Derivate dy/dx with respect to x we get

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = 20 \times 19x^{18} = 380x^{18}$

Question 3

x cos x

Solution:

Let $y = x\ cos\ x$

$\frac{dy}{dx} = x\frac{d}{dx}\cos x + \cos x\frac{d}{dx}x$

$= -x\sin x + \cos x$

Now

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{d}{dx}(x\sin x) + \frac{d}{dx}\cos x$

$= -\left[x\frac{d}{dx}\sin x + \sin x\frac{d}{dx}x\right] - \sin x$

$= -(x\cos x + \sin x) - \sin x$

$= -x\cos x - \sin x - \sin x$

$= -x\ coa\ x - 2\sin x$

$= -(x\cos x + 2\sin x)$

Question 4

log *x*

Solution:

Let y = log x

$\frac{dy}{dx} = \frac{1}{x}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}x^{-1}$

$\frac{d^2y}{dx^2} = (-1)x^{-2} = \frac{-1}{x^2}$

Question 5

x^3 log *x*

Solution:

Let y = x^3 log x

$\frac{dy}{dx} = x^3 \frac{d}{dx} \log x + \log x \frac{d}{dx} x^3$
$= x^3 . \frac{1}{x} + \log x\ (3x^2)$
$= x^2 + 3x^2 \log x$
Now,
$\frac{d^2y}{dx^2} = \frac{d}{dx}\ (x^2 + 3x^2 \log x)$
$= \frac{d}{dx} x^2 + 3 \frac{d}{dx}\ (x^2 \log x)$
$= 2x + 3\left[x^2 \frac{d}{dx} \log x + \log x \frac{d}{dx} x^2\right]$
$= 2x + 3\left(x^2 . \frac{1}{x} + (\log x)\ 2x\right)$
$= 2x + 3\ (x + 2x \log x)$
$= 2x + 3x + 6x \log x$
$= 5x + 6x \log x$
$= x\ (5 + 6 \log x)$

Question 6

$e^x\ sin\ 5x$

Solution:

Let $y = e^x\ sin\ 5x$
$\frac{dy}{dx} = e^x \frac{d}{dx} \sin 5x + \sin 5x \frac{d}{dx} e^x$
$= e^x\ cos\ 5x \frac{d}{dx}\ 5x + \sin 5x\ e^x$
$= e^x \cos 5x\ \times 5 + e^x \sin 5x$
$= e^x\ (5 \cos 5x + \sin 5x)$
Now
$\frac{d^2y}{dx^2} = e^x \frac{d}{dx}\ (5 \cos 5x + \sin 5x)\ + (5 \cos 5x + \sin 5x) \frac{d}{dx} e^x$
$= e^x \{5\ (-\sin x)\ \times 5 + (\cos 5x)\ \times 5\} + (5 \cos 5x + \sin 5x)\ e^x$
$= e^x\ (-25\ sin\ 5x + 5\ cos\ 5\ x + cos\ 5\ x + sin\ 5x)$
$= e^x (10 \cos 5\ x - 24 \sin 5x)$
$= 2e^x\ (5\ cos\ 5x - 12\ sin\ 5x)$

Question 7

$e^{6x}\ cos\ 3x$

Solution:

Let $y = e^{6x}\ cos\ 3x$
$\frac{dy}{dx} = e^{6x} \frac{d}{dx} \cos 3\ x + \cos 3x \frac{d}{dx}\ e^{6x}$

$= e^{6x}\left(-\sin 3x\right)\frac{d}{dx}(3x) + \cos 3x\, e^{6x}\frac{d}{dx}(6x)$
$= -e^{6x}\sin 3x \times 3 + \cos 3x\, e^{6x} \times 6$
$= e^{6x}(-3\sin 3x + 6\cos 6x)$
Now,
$\frac{d^2y}{dx^2} = e^{6x}\frac{d}{dx}(-3\sin 3x + 6\cos 3x) + (-3\sin 3x + 6\cos 3x)\frac{d}{dx}e^{6x}$
$= e^{6x}(-3\cos x \times 3 - 6\sin 3x \times 3) + (-3\sin 3x + 6\cos 3x)e^{6x} \times 6$
$= e^{6x}(-9\cos 3x - 18\sin 3x - 18\sin 3x + 36\cos 3x)$
$= 9e^{6x}(3\cos 3x - 4\sin 3x)$

Question 8

$\tan^{-1} x$

Solution:

Let $y = \tan^{-1} x$
$\frac{dy}{dx} = \frac{1}{1+x^2}$
$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{1+x^2}\right)$
$= \frac{(1+x^2)\frac{d}{dx}(1) - 1\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$
$= \frac{(1+x^2)\times 0 - 2x}{(1+x^2)^2}$
$= \frac{-2x}{(1+x^2)^2}$

Question 9

$\log(\log x)$

Solution:

Let $y = \log(\log x)$
$\frac{dy}{dx} = \frac{1}{\log x}\frac{d}{dx}\log x$
$\left[\because \frac{d}{dx}\log f(x) = \frac{1}{f(x)}\frac{d}{dx}f(x)\right]$
$= \frac{1}{\log x}\cdot\frac{1}{x} = \frac{1}{x\log x}$
$\Rightarrow \frac{d^2y}{dx^2} = \frac{(x\log x)\frac{d}{dx}(1) - 1\frac{d}{dx}(x\log x)}{(x\log x)^2}$
$= \frac{(x\log x)(0) - \left[x\frac{d}{dx}\log x + \log x\frac{d}{dx}x\right]}{(x\log x)^2}$

$= \frac{\left[x \frac{1}{x} + \log x \times 1\right]}{(x \log x)^2}$

$= \frac{[1 + \log x]}{(x \log x)^2}$

Question 10

sin (log x)

Solution:

Let $= y = \sin(\log x)$

$\frac{dy}{dx} = \cos(\log x) \frac{d}{dx} (\log x)$

$= \cos(\log x).\frac{1}{x}$

$= \frac{\cos(\log x)}{x}$

Now,

$\frac{d^2y}{dx^2} = \frac{x \frac{d}{dx}\cos(\log x) - \cos(\log x) \frac{d}{dx} x}{x^2}$

$= \frac{x\,[-\sin(\log x)] \frac{d}{dx} (\log x) - \cos(\log x) \times 1}{x^2}$

$= \frac{-x \sin\ (\log x)\frac{1}{x} - \cos\ (\log x)}{x^2}$

$= \frac{-[\sin\ (\log x) + \cos\ (\log x)]}{x^2}$

Question 11

if $y = 5\ cos\ x - 3\ sin\ x$. prove that $\frac{d^2y}{dx^2} + y = 0$

Solution:

Let $y = 5\ cos\ x - 3\ sin\ x$ .... (1)

$\frac{dy}{dx} = -5 \sin x - 3 \cos x$

$\frac{d^2y}{dx^2} = -5 \cos x + 3 \sin x$

$= -(5 \cos x - 3 \sin x) = -y$ [from (1)]

$\frac{d^2y}{dx^2} = -y$

$\frac{d^2y}{dx^2} + y = 0$

Question 12

If $y = cos^{-1} x \, find \, \frac{d^2y}{dx^2}$ in terms of y alone

Solution:

Given $y = cos^{-1} x$
Or x = cos y ……….. (1)
$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$
$= \frac{-1}{\sqrt{1-cos^2 y}}$ [From (1)]
$= \frac{-1}{\sqrt{1-x^2}} = \frac{-1}{\sin y} = -cosex\, y$ …………. (2)
Now,
$\frac{d^2y}{dx^2} = \frac{d}{dx}(cosec\, y)$
$= -\left[- cosec\, y \cot y \, \frac{dy}{dx}\right]$
$= cosec\, y \cot y \,(-cosec\, y)$
$= -cosec^2\, y\, cot\, y$ [From (2)]

Question 13

If $y = 3 \cos (\log x) + 4 \sin (\log x)$ show that $x^2 y_2 + xy_1 \; y = 0$

Solution:

Given function is
$y = 3 \cos (\log x) + 4 \sin (\log x)$ ………. (1)
Derivate with respect to x, we get
$\frac{dy}{dx} = y_1 = 3 \sin(\log x) \frac{d}{dx} \log x + 4 \cos(\log x) \frac{d}{dx} \log x$
$y_1 = 3 \sin(\log x) \frac{1}{x} + 4 \cos(\log x) \frac{1}{x}$
$= \frac{1}{x} [- 3 \sin(\log x) + 4 \cos(\log x)]$
$xy_1 = -3\, sin\, (\log x) + 4 \cos(\log x)$
Now derivate above equation once again
$\frac{d}{dx}(xy_1) = -3 \cos(\log x) \frac{d}{dx} \log x - 4 \sin(\log x) \frac{d}{dx} \log x$
$x \frac{d}{dx}(y_1) + y_1 \frac{d}{dx} x = 3 \cos(\log x) \frac{1}{x} - 4\, sin\, (\log x) \frac{1}{x}$
$xy_2 + y_1 = -\frac{[3 \cos(\log x) + 4 \sin(\log x)]}{x}$
$x\,(xy_2 + y_1) = -[3 \cos(\log x) + 4 \sin(\log x)]$
$x\,(xy_2 + y_1) = -y$ [Using equation (1)]
This implies, $x^2 y_2 + xy_1 + y = 0$
Hence proved.

Question 14

If $y = Ae^{mx} + Be^{nx}$, show that
$\frac{d^2y}{dx^2} = (m+n)\frac{dy}{dx} + nmy = 0$

Solution:

To prove $\frac{d^2y}{dx^2} = (m+n)\frac{dy}{dx} + nmy = 0$
$y = Ae^{mx} + Be^{nx}$ ………… (1)
$\frac{dy}{dx} = Ae^{mx}\frac{d}{dx}(mx)Be^{nx}\frac{d}{dx}(nx)\left[\because \frac{d}{dx}e^{f(x)} = e^{f(x)}\frac{d}{dx}f(x)\right]$
$\frac{dy}{dx} = Ame^{mx} + Bne^{nx}$ ………… (2)
Find the derivate equation (2)
$\frac{d^2y}{dx^2} = Ame^{mx}m + Bne^{nx}n$
$= Am^2e^{mx} Bn^2e^{nx}$ ……………. (3)
Now L.H.S. = $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$
(Using equations (1), (2) and (3)
$= Am^2e^{mx} Bn^2e^{nx} - (m+n)Ame^{mx} + Bne^{nx} + mn(Ae^{mx} + Be^{nx})$
$= Am^2e^{mx} + Bn^2e^{nx} - Am^2e^{mx} - Bmne^{nx} + Amne^{nx} - Bn^2e^{nx} + Amne^{nx} + Bmne^{nx}$
$= 0$
$=$ R. H. S.
Hence proved

Question 15

If y = 500 e^{7x} show that
$\frac{d^2y}{dx^2} = 49y.$

Solution:

$y = 500e^{7x} + 600e^{-7x}$ ……….. (1)
$\frac{dy}{dx} = 500e^{7x}(7) + 600e^{-7x}(-7)$
$= 500(7)e^{7x} - 600(7)e^{7x}$
Now,
$\frac{d^2y}{dx^2} = 500(7)e^{7x}(7) - 600(7)e^{-7x}(-7)$
$= 500(49)e^{7x} + 600(49)e^{7x}$
$\Rightarrow \frac{d^2y}{dx^2} = 49[500e^{7x}(7) + 600e^{7x}]$
$= 49_y$ [Using equation (1)]

$\Rightarrow \frac{d^2y}{dx^2} = 49_y$
$\Rightarrow$ Hence proved

Question 16

If $e^y\ (x+1) = 1$ show that
$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

Solution:

Given $e^y\ (x+1) = 1$
So, $e^y\ = \frac{1}{x+1}$
Taking log on both sides we have
$\log\ e^y\ = \log\frac{1}{x+1}$
$y \log e = \log = 1 - \log(x+1)$
$y = \ -\log(x+1)$
$\frac{dy}{dx} = -\ \frac{1}{x+1}\ \frac{d}{dx}\ (x+1)$
$= \frac{1}{x+1} = \ (x+1)^{-1}$
Again
$\frac{d^2y}{dx^2} = \ -(-1)\ (x+1)^{-2}\ \frac{d}{dx}\ (x+1)$
$\left[\because\ \frac{d}{dx}\ \{f\ (x)\}^n\ = n\{f\ (x)\}^{n-1}\ \frac{d}{dx}\ f\ (x)\ \right]$
So, $\frac{d^2y}{dx^2} = \ \frac{1}{(x+1)^2}$
Now L.H.S. = $\frac{d^2y}{dx^2} = \ \frac{1}{(x+1)^2}$
And R.H.S. $\left(\frac{dy}{dx}\right)^2\ = \left(\frac{-1}{X+1}\right)^2\ \frac{1}{(x+1)^2}$
L.H.S. = R.H.S.
Hence proved.

Question 17

If $y = \ (tan^{-1}\ x)^2$ show that
$(x^2+1)^2\ y_2 + \ 2x\ (x^2+\ 1)\ y_1 = 2$

Solution:

Given $y = \ (tan^{-1}\ x)^2$ .... (1)
Respect y_1 as first derivative and y_2 as second derivative of a function.
$y_1 = 2\ (tan^{-1}\ x)\ \frac{d}{dx}\ tan^{-1}\ x$

$\left[\because \frac{d}{dx}\{f(x)\}^n = n\{f(x)\}^{n-1}\frac{d}{dx}f(x)\right]$

And $y_1 = 2(tan^{-1}x)\ \frac{1}{1+x^2}$

$= \frac{2tan^{-1}x}{1+x^2}$

So, $(1+x^2)\,y_1 = 2\,tan^{-1}x$

Again differentiating both side with respect to x

$(1+x^2)\frac{d}{dx}y_1 + y_1\frac{d}{dx}(1+x^2) = 2.\frac{1}{1+x^2}$

$(1+x^2)y_1 + y_1\,2x\,\frac{2}{1+x^2}$

$(1+x^2)^2\,y_1 + 2xy_1\,(1+x^2) = 2$

Hence proved

Exercise 5.8

Question 1

Verify Rolle's Theorem for $f(x) = x^2 + 2x - 8, x \in [-4, 2]$.

Solution:

Given function is $f(x) = x^2 + 2x - 8, x \in [-4,2]$

(a) f(x) is a polynomial and polynomial function is always continuous.

So, function is continuous in [-4, 2].

(b) $f(x) = 2x + 2, f(x)$exists in $[-4,2]$so derivable

(c) $f(-4) = 0\ and\ f(2) = 0$

$f(-4) = f(2)$

All three conditions of Rolle's Theorem are satisfied.

Therefore, there exists, at least one $c \in (-4,2)$ such that $f`(c) = 0$

Which implies, 2c + 2 = 0 or c = -1.

Question 2

Examine if Rolles/ theorem is applicable to any of the following functions. Can you say something about the converse of Rolle's Theorem from these examples?

I. $f(x) = [x]$ **for** $x \in [5, 9]$
II. $f(x) = [x]$ **for** $x \in [-2, 2]$
III. $f(x) = x^2\,[x]$ **for** $x \in [1, 2]$

Solution:

I. Function is greatest integer function.
Given function is not differentiable and continuous
Hence Rolle's theorem is not applicable here.
II. Function is greatest integer function.
Given function is not differentiable and continuous.

Hence Rolle's theorem is not applicable.

III. $f\ (x) =\ (2)^2 - \Rightarrow f\ (\mathrm{I}) =\ (1)^2 -\ 1 = 1 - = 0$

$f\ (2) = (2)^2 -\ 1 = 4 - 1 = 3\ \therefore f\ (1) \neq f\ (2)$

Rolle`s theorem is not applicable.

Question 3

If $f\ :\ [-5, 5] \rightarrow$ R is differentiable function are if f' (x) does not vanish anywhere, then prove that $f\left(-5\right) \neq f\ (5)$

Solution:

As per Rolle`s theorem if

a) f is continuous is $[a, b]$
b) f is continuous is $[a, b]$
c) f is continuous is $[a, b]$

Then $f`(c) =\ 0. c\ \in (a, b)$

It is given that f is continuous and derivable but $f^{'}(c) \neq 0$

$\Rightarrow f\ (a) \neq f\ (b)$

$\Rightarrow f\ (-5) \neq f\ (5)$

Question 4

Verify mean value Theorem if

$f\ (x) =\ x^2 -\ 4x - 3$

In the interval $[a, b]$ where a = 1 and b = 4

Solution:

(a) f(x) is a polynomial

So, functions is continuous in [1, 4] as polynomial functions is always continuous.

(b) $f\ (x) =\ 2x - 4, f\ (x)$ Exists in [1, 4] hence derivable.

Both the conditions of the functions are satisfied, so there exists, at least one $c\ \epsilon\ (1,4)$ such that

$\frac{f\ (4) - f\ (1)}{4-1}\ f^{'}(c)$

$\frac{-3-(-6)}{3} = 2\ c - 4$

$1 = 2c - 4$

$c = \frac{5}{2}$

Question 5

Verify mean value Theorem if $f\ (x) =\ x^3 -\ 5x^2 -\ 3x$ in the interval [a,b] where a = 1 and b = 3 find all c $\in (1, 3)$ for which $f^{'}(c) = 0$.

Solution:

(a) Function is a polynomial as polynomial function is always continuous. So continuous in [1,3]
(b) $f(x) = 3x^2 - 10x$ $f(x)$ Exists in [1, 3] hence derivable
Conditions of MVT theorem are satisfied so there exists at least one $c \in (1,3)$ such that
$\frac{f(3)-f(1)}{3-1} = f'(c)$
$\frac{-21-(-7)}{2} = 3c^2 - 10c$
$-7 = 3c^2 - 10c$
$3c^2 - 7c - 3c + 7 = 0$
$c(3c-7) - (3c-7) = 0$
$(3c-7)(c-1) = 0$
$(3c-7) = 0$ or $(c-1) = 0$
$3c = 7$ or $c = 1$
$c = \frac{7}{3}$ or $c = 1$
Only $c = \frac{7}{3} \epsilon (1,3)$
As, $f(1) \neq f(3)$ therefore the value of c does not exist such that $f(c) = 0$

Question 6

Examine the applicability of Mean value Theorem for all the functions be given below: [Note for students: check exercise 2]

I. $f(x) = [x]$ for $x \in [5,9]$
II. $f(x) = [x]$ for $x \in [-2,2]$
III. $f(x) = x^2 - 1[x]$ for $x \in [1,2]$

Solution:

According to means value theorem:
For a function $f : [a,b] \to R$, if
a) f is continuous on (a,b)
b) f is differentiable on (a,b)
Then there exist some $c \epsilon (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$
(i) $f(x) = [x]$ for $x \epsilon [5,9]$
Given function f (x) is not continuous at $x = 5$ and $x = 9$
Therefore,
$f(x)$ is not continuous at [5,9]
Now let n be an integer such that $n \epsilon [5,9]$

$$\therefore \text{L.H.L.} = \lim_{h\to0^-} \frac{f(n+h) - f(n)}{h} = \lim_{h\to0^-} \frac{(n+h)-(n)}{h} = \lim_{h\to0^-} \frac{n-1-n}{n} = \lim_{h\to0^-} \frac{-1}{h} = \infty$$

And R. H. L. $= \lim_{h\to 0^-} \frac{f(n+h) - f(n)}{h} = \lim_{h\to 0^-} \frac{(n+h)-(n)}{h} = \lim_{h\to 0^-} \frac{n-n}{n} = \lim_{h\to 0^-} 0 = 0$

Since, L.H.L. ≠ R.H.L.,

Therefore f is not differentiable at [5, 9].

Hence Mean Value Theorem is not applicable for this function.

(iii) $f(x) = [x]$ For $x \epsilon [-2, 2]$

Given function $f(x)$ is not continuous at $x = -2$ and $x = 2$

Therefore

$f(x)$ is not continuous at [-2,2]

Now let n be an integer such that $n \epsilon$ [-2,2]

∴ L. H. L. $= \lim_{h\to 0^-} \frac{f(n+h) - f(n)}{h} = \lim_{h\to 0^-} \frac{(n+h)-(n)}{h} = \lim_{h\to 0^-} \frac{n-1-n}{n} = \lim_{h\to 0^-} \frac{-1}{h} = \infty$

And R. H. L. $= \lim_{h\to 0^-} \frac{f(n+h) - f(n)}{h} = \lim_{h\to 0^-} \frac{(n+h)-(n)}{h} = \lim_{h\to 0^-} \frac{n-n}{n} = \lim_{h\to 0^-} 0 = 0$

Since, L.H.L. ≠ R.H.L

Therefore f is not differentiable at [-2, 2].

Hence Mean Value Theorem is not applicable for this function.

(iii) $f(x) = x^2 \; for \; x \in [1,2] \ldots\ldots\ldots..(1)$

Here, $f(x)$ is a polynomial function.

Therefore, , $f(x)$ is continuous and derivable on the real line.

Hence, $f(x)$ is continuous in the closed interval [1, 2] and derivable in open interval (1, 2).

Therefore, both conditions of Mean Value Theorem are satisfied.

Now, from equation (1), we have

$f(x) = 2x$

$f'(c) = 2c$

Again From equation (1):

$f(a) = f(1) = (1)^2 - 1 = 1 - 1 = 0$

And $f(b) = f(2) = (2)^2 - 1 = 4 - 1 = 3$

Therefore

$f\ c = \frac{f(b) - f(a)}{b - 1}$

$2c = \frac{3-0}{2-1}$

$c = \frac{3}{2} \epsilon (1,2)$

Therefore mean value Theorem is verified.

Miscellaneous Exercise

Differentiate with respect to x the functions in Exercises 1 to 11.

Question 1

$(3x^2 - 9x + 5)^0$

Solution:

Consider $(3x^2 - 9x + 5)^0$

$\frac{dy}{dx} = 9\,(3x^2 - 9x + 5)^8 \frac{d}{dx}\,(3x^2 - 9x + 5)$

$\left[\therefore \frac{d}{dx}\,\{f(x)\}^n = n\,\{f(x)\}^{n-1} \frac{d}{dx}\,f\,(x)\right]$

$\frac{dy}{dx} = 9\,(3x^2 - 9x + 5)^8\,[3\,(2x) - 9\,(1) + 0]$

$\frac{dy}{dx} = 27\,(3x^2 - 9x + 5)^8\,[2x - 3]$

Question 2

$sin^3\,x + cos^6\,x$

Solution:

Consider y $= sin^3\,x + cos^6\,x$

or y = $(\sin x)^3 + (\cos)^6$

$\frac{dy}{dx} = 3\,(\sin x)^2\,\frac{d}{dx} \sin x + 6\,(\cos x)^5\,\frac{d}{dx} \cos x$

$\frac{dy}{dx} = 3\,sin^2\,x \cos x - 6\,cos^5\,x \sin x$

$= 3\,sin\,x\,cos\,x\,(sin\,x - 2\,cos^4\,x)$

Question 3

$(5\,x)^{3\,cos\,2\,x}$

Solution:

consider y $= (5\,x)^{3\,cos\,2\,x}$

Taking both the sides we get

log y = $\log\,(5\,x)^{3\,cos\,2\,x}$

$\log y = 3 \cos 2\,x \log(5x)$

Derivate above function:

$\frac{d}{dx} \log y = 3\,\frac{d}{dx}\,\{\cos 2\,x \log(5x)\}$

$\frac{1}{y}\frac{dy}{dx} = 3\left[\cos 2x\,\frac{d}{dx} \log(5x) + \log(5x) \log \frac{d}{dx} \cos 2x\right]$

$$\frac{1}{y}\frac{dy}{dx} = 3\left[\cos 2x\ \frac{1}{5x}\ \frac{d}{dx}\ 5x + \log(5x)(-\sin 2x)\ (-\sin 2x)\ \frac{d}{dx}\ 2x\right]$$
$$\frac{1}{y}\frac{dy}{dx} = 3\left[\cos 2x\ \frac{1}{5x}\ .5 - 2\sin 2x\log(5x)\right]$$
$$\frac{dy}{dx} = 3y\left[\frac{\cos 2x}{x} - 2\sin 2\,x\log(5x)\right]$$
$$\frac{dy}{dx} = 3\ (5\,x)^{3\cos 2\,x}\left[\frac{\cos 2x}{x} - 2\sin 2\,x\log(5x)\right]$$ [Using value of y]

Question 4

$\boldsymbol{sin^{-1}\left(x\sqrt{x}\right), 0 \le x \le 1}$

Solution:

Consider $sin^{-1}\left(x\sqrt{x}\right)$

Or y = $sin^{-1}\left(x^{\frac{3}{2}}\right)$

Apply derivation:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(x^{\frac{3}{2}}\right)^2}}\ \frac{d}{dx}\ x^{\frac{3}{2}}$$
$$= \frac{1}{\sqrt{1-x^3}}.\frac{3}{2}\ x^{\frac{1}{2}}$$
$$= \frac{3}{2}\sqrt{\frac{x}{1-x^3}}$$

Question 5

$\frac{\boldsymbol{cos^{-1}\frac{x}{2}}}{\boldsymbol{\sqrt{2x+7}}}, -2 < x < 2$

Solution:

Consider $\frac{cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}$

Apply derivation:

$$\frac{dy}{dx} = \frac{\sqrt{2x+7}\,\frac{d}{dx}\,cos^{-1}\frac{x}{2} - cos^{-1}\frac{x}{2}\frac{d}{dx}\sqrt{2x+7}}{\left(\sqrt{2x+7}\right)^2}$$ [Using Quotient Rule]

$$\frac{dy}{dx} = \frac{\sqrt{2x+7}\left(\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\right)\frac{d}{dx}\frac{x}{2} - \left(cos^{-1}\frac{x}{2}\right)\frac{1}{2}(2x+)^{\frac{-1}{2}}\frac{d}{dx}(2x+7)}{\left(\sqrt{2x+7}\right)^2}$$

$$\frac{dy}{dx} = \frac{-\sqrt{2x+7}.\frac{2}{\sqrt{4-x^2}}.\frac{1}{2} - \frac{1}{2}\frac{cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}}{(2x+7)}$$

$$= \frac{\left[\frac{\sqrt{2x+7}}{\sqrt{4-x^2}} + \frac{cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}\right]}{(2x+7)}$$

$$\frac{dy}{dx} = -\left[\frac{2x+7+\sqrt{4-x^2}\,cos^{-1}\frac{x}{2}}{\sqrt{4-x^2}\,\sqrt{2x+7}\,(2x+7)}\right]$$

$$= -\left[\frac{2x+7+\sqrt{4-x^2}\cos^{-1}\frac{x}{2}}{\sqrt{4-x^2})\,(2x+7)^{\frac{3}{2}}}\right]$$

Question 6

$$\boldsymbol{cot^{-1}\left[\frac{\sqrt{1+sin\,x}+\sqrt{1-sin\,x}}{\sqrt{1+sin\,x}-\sqrt{1-sin\,x}}\right]\ 0 < x < \frac{\pi}{2}}$$

Solution:

Consider y $= cot^{-1}\left[\frac{\sqrt{1+sin\,x}+\sqrt{1-sin\,x}}{\sqrt{1+sin\,x}-\sqrt{1-sin\,x}}\right]\ 0 < x < \frac{\pi}{2}$ .... (i)

Reduce the functions into simplest for,

$\sqrt{1+sin\,x} = \sqrt{cos^2\frac{x}{2} + sin^2\frac{x}{2} + 2\sin\frac{x}{2}\,cos\frac{x}{2}}$

$= \sqrt{\left(cos\frac{x}{2} + sin\frac{x}{2}\right)^2} = cos\frac{x}{2} + sin\frac{x}{2}$

And $\sqrt{1-sin\,x} = \sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\,cos\frac{x}{2}}$

$= \sqrt{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}\ \ \cos\frac{x}{2} - \sin\frac{x}{2}$

Now we are available with the equation below

$$y = \cot^{-1}\left(\frac{\cos\frac{x}{2}+\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}+\sin\frac{x}{2}}\right)$$
$$= \cot^{-1}\left(\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right)$$
$$y == cot^{-1}\left(cos\ \frac{x}{2}\right)$$
$$= \frac{x}{2}$$

Applying derivation

$$\frac{dy}{dx} = \frac{1}{2}(1) = \frac{1}{2}$$

Question 7

$(\log x)^{\log x}, x > 1$

Solution:

Consider $y = (\log x)^{\log x}, x > 1$

Taking log both sides

$\log y = \log (\log x)^{\log x} = \log x \log x (\log x)$

Applying derivation

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(\log x \log(\log x))$$
$$\frac{1}{y}\frac{dy}{dx} = \log x\frac{d}{dx}\log(\log x) + \log(\log x)\frac{d}{dx}\log x$$
$$\frac{1}{y}\frac{dy}{dx} = \log x\frac{1}{\log x}\frac{d}{dx}(\log x) + \log(\log x)\frac{1}{x}$$
$$= \frac{1}{x} + \frac{\log(\log x)}{x}$$
$$\frac{dy}{dx} = y\left(\frac{1+\log(\log x)}{x}\right)$$
$$= (\log x)^{\log x}\left(\frac{1+\log(\log x)}{x}\right)$$

Question 8

$cos\ (a\ cos\ x + b\ sin\ x)$ From some constants a and b.

Solution:

Consider $y = cos\ (a\ cos\ x + b\ sin\ x)$ for some constants a and b

Apply derivation

$\frac{dy}{dx} = -\sin(a\cos x + b\sin x)\,\frac{dy}{dx}(a\cos x + b\sin x)$

$\frac{dy}{dx} = -\sin(a\cos x + b\sin x)\,(-a\sin x + b\cos x)$

$\frac{dy}{dx} = -(-a\sin x + b\cos x)\,\sin(a\cos x + b\sin x)$

$\frac{dy}{dx} = (-a\sin x - b\cos x)\,\sin(a\cos x + b\sin x)$

Question 9

$(\sin x - \cos x)^{\sin x - \cos x}\ \frac{\pi}{4} < x < \frac{3\pi}{4}$

Solution:

Consider $(\sin x - \cos x)^{\sin x - \cos x}$ ………… (i)

Applying derivation:

$\log y = \log(\sin x - \cos x)^{\sin x - \cos x}$

$= (\sin x - \cos x)\log(\sin x - \cos x)$

Applying derivation

$\frac{d}{dx}\log y = (\sin x - \cos x)\,\frac{d}{dx}(\sin x - \cos x) + \log(\sin x - \cos x)\,\frac{d}{dx}(\sin x - \cos x)$

$\frac{1}{y}\frac{d}{dx} = (\sin x - \cos x)\,\frac{1}{(\sin x - \cos x)}\,\frac{d}{dx}(\sin x - \cos x) + \log(\sin x - \cos x).(\cos x + \sin x)$

$\frac{1}{y}\frac{d}{dx} = (\cos x + \sin x) + (\sin x + \cos x)\log(\sin x - \cos x)$

$\frac{1}{y}\frac{d}{dx} = (\cos x + \sin x)\,[1 + \log(\sin x + \cos x)]$

$\frac{d}{dx} = y\,(\cos x + \sin x)\,[1 + \log(\sin x + \cos x)]$

$\frac{d}{dx} = (\sin x - \cos x)^{\sin x - \cos x}\,(\cos x + \sin x)\,[1 + \log(\sin x + \cos x)]$

Question 10

$x^x + x^a + a^x + a^a$ For some fixed a> 0 and x >0

Solution:

$y = x^x + x^a + a^x + a^a$

Applying derivation

$$\frac{dy}{dx} = \frac{d}{dx}\, x^x\, \frac{d}{dx}\, x^a + \frac{d}{dx} a^x + \frac{d}{dx} a^a$$

$= \frac{d}{dx}\, x^x + a x^{a-1} + a^x \log a + 0$ ………….. (i)

First term form equation (i):

$\frac{d}{dx}\,(x^x)$, Consider $u = x^x$

$\log u = \log x^x = x \log x$

$$\frac{d}{dx} \log u = \frac{d}{dx}\,(x \log x)$$

$$\frac{1}{u}\frac{du}{dx} = x\,\frac{d}{dx} \log x + \log x\,\frac{d}{dx}\, x$$

$$\frac{1}{u}\frac{du}{dx} = x\,\frac{1}{x} + \log x\; 1$$

$= 1 + \log x$

This implies,

$$\frac{du}{dx} = u\,(1 + \log x)$$

Substitute value of u back

$\frac{d}{dx} x^x = x^x\,(1 + \log x)$ ………………. (ii)

Using equation (ii) in (i) we have

$$\frac{d}{dx} = x^x (1 + \log x) a x^{a-1} + a^x \log x$$

Question 11

x^{x^2} -3+ $(x-3)^{x^2}$ for x > 3

Solution:

Consider $y = x^{x^2}$ -3+ $(x-3)^{x^2}$ for x > 3

Put $u = x^{x^2}$ -3 $v = (x-3)^{x^2}$

$\frac{du}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ………. (i)

Now $u = x^{x^2}$ -3

$\log u = \log x^{x^2} = (x^2 - 3) \log x$

$$\frac{1}{u}\,\frac{du}{dx}\,(x^2 - 3)\,\frac{d}{dx} \log x + \log x\,\frac{d}{dx}\,(x^2 - 3)$$

$$= (x^2 - 3)\,\frac{1}{x} + \log x\,(2x - 0)$$

$$\frac{1}{u}\,\frac{du}{dx} = \frac{x^2 - 3}{x} + 2x \log x$$

$$\frac{du}{dx} = u\left(\frac{x^2 - 3}{x} + 2x \log x\right)$$

$\frac{du}{dx} = x^{x^2} - 3\,\left(\frac{x^2 - 3}{x} + 2x \log x\right)$ …………. (ii)

Again $v = (x-3)^{x^2}$
$\log x = \log(x-3)^{x^2}$
$= x^2 \log(x-3)$
$\frac{1}{v}\frac{dv}{dx} = x^2 \frac{d}{dx}\log(x-3) + \log(x-3)\frac{d}{dx}x^2$
$= x^2 \frac{1}{x-3}\frac{d}{dx}(x-3) + \log(x-3)\,2x$
$\frac{1}{v}\frac{dv}{dx} = \frac{x^2}{x-3} + 2x\log(x-3)$
$\frac{dv}{dx} = v\left[\frac{x^2}{x-3} + 2x\log(x-3)\right]$
$\frac{dv}{dx} = (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x\log(x-3)\right]$ ………. (iii)
Using equation (ii) and (iii) in eq. (i) we have
$\frac{dv}{dx} = x^{x^2-3}\left(\frac{x^2-3}{x} + 2x\log x\right) + (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x\log(x-3)\right]$

Question 12

Find $\frac{dy}{dx}$ if $y = 12(1-\cos t)$ and $x = 10(t-\sin t)$, $\frac{\pi}{2} < t < \frac{\pi}{2}$

Solution:

Given expression are $y = 12(1-\cos t)$ and $x = 10(t-\sin t)$
$\frac{dy}{dt} = 12\frac{d}{dt}(1-\cos t) = 12(0+\sin t) = 12\sin t$
And $\frac{dx}{dt} = 10\frac{d}{dt}(1-\cos t)$
$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{12\sin t}{10(1-\cos t)}$
$= \frac{6}{5}\frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\sin^2\frac{t}{2}}$
$= \frac{6}{5}\cdot\frac{\cos\frac{t}{2}}{\sin\frac{t}{2}} = \frac{6}{5}\cot\frac{t}{2}$

Question 13

Find $\frac{dy}{dx}$ if $y = \sin^{-1}x + \sin^{-1}\sqrt{x-x^2}$, $- \le x \le 1$.

Solution:

Given expression is $y = sin^{-1} x + sin^{-1} \sqrt{x - x^2}$

Apply derivation:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} + \frac{1}{\sqrt{1 \left(\sqrt{1 - x^2}\right)^2}} \frac{d}{dx} \sqrt{x - x^2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} + \frac{1}{\sqrt{1 \left(\sqrt{1 - x^2}\right)^2}} \frac{1}{2}(1 - x^2)^2 \frac{d}{dx} (1 - x^2)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} + \frac{1}{\sqrt{1 - 1 + x^2}} \frac{1}{2\sqrt{1 - x^2}} (-2x)$$

$$\frac{1}{\sqrt{1 - x^2}} + \frac{1}{\sqrt{x^2}} \left(\frac{-x}{\sqrt{1 - x^2}}\right)$$

Which implies?

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} - \frac{x}{x\sqrt{1 - x^2}}$$

$$= \frac{1}{\sqrt{1 - x^2}} - \frac{1}{\sqrt{1 - x^2}} = 0$$

Therefore, dy/dx = 0

Question 14

If $x \sqrt{1 + y} + y \sqrt{1 + x} = 0$ for $-1 < x < 1$.

Prove that $\frac{dy}{dx} = \frac{-1}{(1+x)^2}$

Solution:

Given expression is $x \sqrt{1 + y} + y \sqrt{1 + x} = 0$

$\sqrt{1 + y} = -y\sqrt{1 + x}$

Squaring both sides:

$x^2 (1 + y) = y^2 (1 + x)$

$x^2 + x^2 y = y^2 + y^2 x$

$x^2 - y^2 = -x^2 y + y^2 x$

$(x - y)(x + y) = -xy(x - y)$

$x + y = -xy$

$\Rightarrow y(1 + x) = -x$

$\Rightarrow y = \frac{-x}{1+x}$

Apply derivation

$$\frac{dy}{dx} = -\frac{(1+x)\frac{d}{dx}x - x\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$= \frac{(1+x)1 - x\,1}{(1+x)^2}$$

$= \frac{1}{(1+x)^2}$

Hence Proved

Question 15

$if\ (x-a)^2 + (y-b)^2 = c^2$ For some c> 0 prove that

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

Is a constant independent of a and b

Solution:

Give expression is $(x-a)^2 + (y-b)^2 = c^2$ .... (1)

Apply derivation:

$2\,(x-a) + 2\,(y-b)\frac{dy}{dx} = 0$

$2\,(x-a) = -2\,(y-b)\frac{dy}{dx}$

$\frac{dy}{dx} = -\left(\frac{x-a}{y-b}\right)$ .... (2)

Again $\frac{d^2y}{dx^2} = \frac{-\left[(y-b).1-(x-a)\frac{dy}{dx}\right]}{(y-b)^2}$

$\frac{d^2y}{dx^2} = \frac{-\left[(y-b).1-(x-a)\left(\frac{-(x-a)}{y-b}\right)\right]}{(y-b)^2}$ [Using equation (2)]

$$\frac{d^2y}{dx^2} = \frac{-\left[(y-b)+\left(\frac{(x-a)^2}{y-b}\right)\right]}{(y-b)^2}$$

$$= \frac{-\left[(y-b)^2 + (x-a)^2\right]}{(y-b)^3}$$

$= \frac{-c^2}{(y-b)^3}$ .... (3)

Put values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given, we get

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

$$=\frac{\left[1+\frac{(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{\frac{-c^2}{(y-b)^3}}$$

$$=\frac{\left((y-b)^2+(x-a)^2\right)^{\frac{3}{2}}}{(y-b)^3}\times\frac{(y-b)^3}{-c^2}=\frac{(c^2)^{\frac{3}{2}}}{-c^2}=-c \text{ (Constant value)}$$

Question 16

If $\cos y = x\cos(a+y)$ with $\cos a \neq \pm 1$. *prove that* $\frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$

Solution:

Given expression is $\cos y = x\cos(a+y)$

$$x=\frac{\cos y}{\cos(a+y)}$$

Apply derivative w.r.t. y

$$\frac{dx}{dy}=\frac{d}{dy}\left(\frac{\cos y}{\cos(a+y)}\right)$$

$$\frac{dx}{dy}=\frac{\cos(a+y)\frac{d}{dx}\cos y-\cos y\frac{d}{dx}\cos(a+y)}{\cos^2(a+y)}$$

$$\frac{dx}{dy}=\frac{\cos(a+y)(-\sin y)-\cos y\{-\sin(a+y)\}}{\cos^2(a+y)}$$

$$=\frac{-\cos(a+y)\sin y+\sin(a+y)\cos y}{\cos^2(a+y)}$$

$$\Rightarrow\frac{dx}{dy}=\frac{\sin(a+y-y)}{\cos^2(a+y)}$$

$$=\frac{\sin a}{\cos^2(a+y)}$$

$$\Rightarrow\frac{dx}{dy}=\frac{\cos^2(a+y)}{\sin a}$$ [Take reciprocal]

Question 17

If $x=a(\cos t+t\sin t)$ and $y=a(\sin t-t\cos t)$ find $\frac{d^2y}{dx^2}$

Solution:

Given expression are $x=a(\cos t+t\sin t)$ and $y=a(\sin t-t\cos t)$

$x=a(\cos t+t\sin t)$

Differentiating both sides w.r.t. t

$\frac{dx}{dt} = a\left(-\sin t + \frac{d}{dt}\, t\sin t\right)$

$\frac{dx}{dt} = a\left(-\sin t + t\,\frac{d}{dt}\, t\sin t + \sin t\,\frac{d}{dt}\, t\right)$

$\frac{dx}{dt} = a\,(-\sin t + t\cos t + \sin t)$

$\Rightarrow \frac{dx}{dt} = at\cos t$

And

$y = a\,(\sin t - t\cos t)$

Differentiating both sides w.r.t. t

$\frac{dy}{dt} = a\left(\cos t - \frac{d}{dt}\, t\cos t\right)$

$\frac{dy}{dt} = a\left(\cos t - \left(t\,\frac{d}{dt}\cos t + \cos t\,\frac{d}{dt}\, t\right)\right)$

$\frac{dy}{dt} = a\,(\cos t - (-t\sin t + \cos t))$

$\frac{dy}{dt} = a\,t\sin t$

Now $\frac{dy}{dt} = \frac{dy/dt}{dx/dt} = \frac{at\sin t}{a\,t\cos t} = \frac{\sin t}{\cos t} = \tan t$

Again $\frac{d^2y}{dx^2} = \frac{d}{dx}\tan t = sec^2\, t\,\frac{d}{dx}\, t$

$= sec^2\, t\,\frac{dt}{dx} = sec^2 t\,\frac{1}{at\cos t}$

$= sec^2\, t\,\frac{\sec t}{at} = \frac{sec^3\, t}{at}$

Question 18

If $f(x) = |x|^3$ show that $f''(x)$ exists for all real x find it.

Solution:

Given expression is $f(x) = |x| = \begin{cases} x^3\ if\ x \geq 0 \\ (-x^3), if\ x < 0 \end{cases}$

Step 1: when x < 0

f(x) = -x^3

Differentiate w.r.t. to x,

f’(x) = -3x^2

Differentiate w.r.t. to x,

f’’(x) = -6x, exist for all values of x < 0.

Step 2: When x ≥ 0

f(x) = x^3

Differentiate w.r.t. to x,

f’(x) = 3x^2

Differentiate w.r.t. to x,

f''(x) = 6x, exist for all values of x > 0.

Step 3: When x = 0

$$\lim_{h \to 0^-} \frac{f(0) - f(0+h)}{h} = \lim_{h \to 0^-} \frac{f(0+h) - f(o)}{h} = f'(c)$$

$$f'(x) = \begin{cases} 3x^2 \ if \ x \geq 0 \\ -3x^2 \ if \ x < 0 \end{cases}$$

Now, Check differentiability at x =0

L.H.D at x = 0

$$\lim_{h \to 0^-} \frac{f(0) - f'(0+h)}{h}$$

$$= \lim_{h \to 0^+} \frac{3(0) - (-3(-h)^2)}{h}$$

$$= \lim_{h \to 0^+} \frac{3h^2}{h}$$

As h = 0

= 0

And R.H.D. at x = 0

$$\lim_{h \to 0^-} \frac{f(0) - f'(0+h)}{h}$$

$$= \lim_{h \to 0^+} \frac{f'(h) - f'(0)}{h}$$

$$= \lim_{h \to 0^+} \frac{3(h)^2 - 3(0)^2}{h}$$

$$= \lim_{h \to 0^+} 3h = 0 \ (at \ h = 0)$$

Again L.H.D. at $x = 0$ = R.H.D. at x = 0

This implies, f' (x) exists and differentiable at all real values of x

Question 19

Using mathematical induction prove that $\frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n.

Solution:

Consider p (n) be the given statement

$p(n) = \frac{d}{dx}(x^n) = nx^{n-1}$ ………….. (1)

Step 1: Result is true at n = 1

$p(1) = \frac{d}{dx}(x^1) = (1)x^{1-1} = (1)x^0 = 1$

Which is true as? $\frac{d}{dx}(x) = 1$

Step 2: suppose $p(m)$ is true.

$p(m) = \frac{d}{dx}(x^m) = mx^{m-1}$ …………. (2)

Step 3: Prove that result is true for n = m + 1.

$p\ (m+1) = \frac{d}{dx}\ (mx^{m-1}) = \ (m+1)\ mx^{m-1}$
$x^{m+1} = \ x^1 + \ x^m$
$\frac{d}{dx}\ x^{m+1} = \frac{d}{dx}\ (x. x^m)$
$= x\ \frac{d}{dx}\ x^m + \ x^m\ \frac{d}{dx}\ x$
$= \ x. mx^{m-1} + \ x^m\ (1)$
Therefore, $mx^m + \ x^m = \ x^m\ (m+1)$
$(m+1)\ x^m = \ (m+1)\ x^m$
$(m+1)\ x^{\ (m+1)-1}$
Therefore, $p\ (m+1)$ is true if $p\ (m)$ is true but $p\ (1)$ is true.
Thus by principal of induction p (n) is true for all n ∈ N.

Question 20

Using the fact that $\sin\,(A+b) = \ \sin A \cos B + \cos A \sin B$ and the differentiation, obtain the sum formula for cosines

Solution:

Given expression is $\sin\,(A+b) = \ \sin A \cos B + \cos A \sin B$
Consider A and B as function of t and differentiating both sides w.r.t x,
$\cos(A+B)\left(\frac{dA}{dt}+\frac{dB}{dt}\right) = \sin A\,(-\sin B)\ \frac{dB}{dt} + \cos B\ \left(\cos A\ \frac{dB}{dt}\right) + \cos A \cos B\ \frac{dB}{dt} + \sin B\ (-\sin A)\ \frac{dA}{dt}$
$\Rightarrow \cos(A+B)\left(\frac{dA}{dt}+\frac{dB}{dt}\right) = (\cos A \cos B - \sin A \sin B)\ \left(\frac{dA}{dt}+\frac{dB}{dt}\right)$
$\Rightarrow \cos(A+b) = \ (\cos A \cos B - \sin A \sin B)$

Question 21

Does there exist a function which is continuous everywhere but not differentiable at exactly two points?

Solution:

Consider us consider the function f (x) = $|x| + \ |x-1|$
f is continuous everywhere but is not differentiable at x = 0 and x = 1

Question 22

If $y = \begin{vmatrix} f\ (x) & g\ (x) & h\ (x) \\ l & m & n \\ a & b & c \end{vmatrix}$ Prove that

$$\frac{dy}{dx} = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

Solution:

Given expression is

$$y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

Applying derivative:

$$\frac{dy}{dx} = \begin{vmatrix} \frac{d}{dx} f(x) & \frac{d}{dx} g(x) & \frac{d}{dx} h(x) \\ l & m & n \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ 0 & 0 & 0 \\ a & b & c \end{vmatrix} + y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ 0 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

Question 23

If $y = e^{a\cos^{-1} x}, -1 \leq x \leq 1$, show that $(1 - x^2)\frac{d^2 y}{dx^2} - x\frac{dy}{dx} - a^2 y = 0$

Solution:

Given expression is $e^{a\cos^{-1} x}$

$$\frac{dy}{dx} = e^{a\cos^{-1} x} \frac{d}{dx} a\cos^{-1} x$$

$$= e^{a\cos^{-1} x} a\left(\frac{-1}{\sqrt{1-x^2}}\right)$$

$$= \frac{-ay}{\sqrt{1-x^2}}$$

This implies

$$\left(\frac{dy}{dx}\right)^2 = \frac{a^2 y^2}{1-x^2}$$

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = a^2 y^2$$

Differentiating both sides with respect to x we have

$$(1-x^2)\, 2.\frac{dy}{dx}.\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2(-2x) = 2a^2 y\frac{dy}{dx}$$

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} = a^2 y$$

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - a^2 y = 0$$

Hence proved