Chapter 5
Complex Numbers and Quadratic Equations

Exercise 5.1

Question 1

Express each of the complex number given in the Exercises 1 to 10 in the form a + ib.
(5i) (-3/5i)

Answer:

$(5i)(-3/5i) = 5 \times (-3/5) \times i^2$
$= -3 \times -1$ $[i^2 = -1]$
$= 3$
Hence,
$(5i)(-3/5i) = 3 + i0$

Question 2

$\mathbf{i^9 + i^{19}}$

Answer:

$i^9 + i^{19} = (i^2)^4 . i + (i^2)^9 . i$
$= (-1)^4 . i + (-1)^9 . i$
$= 1 \times i + -1 \times i$
$= i - i$
$= 0$
Hence,
$i^9 + i^{19} = 0 + i0$

Question 3

$\mathbf{i^{-39}}$

Answer:

$i^{-39} = 1/ i^{39} = 1/ i^{4\times9+3} = 1/ (1^9 \times i^3) = 1/ i^3 = 1/ (-i)$ $[i^4 = 1, i^3 = -I \text{ and } i^2 = -1]$
Now, multiplying the numerator and denominator by i we get
$i^{-39} = 1 \times i / (-i \times i)$
$= i/ 1 = i$

Hence,
$i^{-39} = 0 + i$

Question 4

3(7 + *i*7) + *i*(7 + *i*7)

Answer:

3(7 + i7) + i(7 + i7) = 21 + i21 + i7 + i2 7
= 21 + i28 – 7 [i2 = -1]
= 14 + i28
Hence,
3(7 + i7) + i(7 + i7) = 14 + i28

Question 5

(1 – i) – (–1 + *i*6)

Answer:

(1 – i) – (–1 + *i*6) = 1 – *i* + 1 - *i*6
= 2 – i7
Hence,
(1 – *i*) – (–1 + *i*6) = 2 – *i*7

Question 6

$\left(\frac{1}{5} + i\,\frac{2}{5}\right) - \left(4 + i\,\frac{5}{2}\right)$

Answer:

$\left(\frac{1}{5} + i\,\frac{2}{5}\right) - \left(4 + i\,\frac{5}{2}\right)$
$= \frac{1}{5} + \frac{2}{5}i - 4 - \frac{5}{2}i$
$= \left(\frac{1}{5} - 4\right) + i\left(\frac{2}{5} - \frac{5}{2}\right)$
$= \frac{-19}{5} + i\left(\frac{-21}{10}\right)$
$= \frac{-19}{5} - \frac{21}{10}i$
Hence,
$\left(\frac{1}{5} + i\,\frac{2}{5}\right) - \left(4 + i\,\frac{5}{2}\right) - \frac{-19}{5} - \frac{21}{10}i$

Question 7

$\left[\left(\frac{1}{3}+i\ \frac{7}{3}\right)+\left(4+\ i\ \frac{1}{3}\right)\right]-\left(-\frac{4}{3}+\ i\ \right)$

Answer:

$\left[\left(\frac{1}{3}+i\ \frac{7}{3}\right)+\left(4+\ i\ \frac{1}{3}\right)\right]-\left(\frac{-4}{3}+\ i\ \right)$

$=\frac{1}{3}+\ \frac{7}{3}i+\ 4+\frac{1}{3}\ i\ +\frac{4}{3}-\ i$

$=\left(\frac{1}{3}+4+\ \frac{4}{3}\right)+\ i\ \left(\frac{7}{3}+\ \frac{1}{3}-1\right)$

$=\frac{17}{3}+\ i\ \frac{5}{3}$

Hence

$\left[\left(\frac{1}{3}+i\ \frac{7}{3}\right)+\left(4+\ i\ \frac{1}{3}\right)\right]-\left(-\frac{4}{3}+\ i\ \right)=\frac{17}{3}+\ i\ \frac{5}{3}$

Question 8

$(1 - i)^4$

Answer:

$(1 - i)^4 = [(1 - i)^2]^2$

$= [1 + i^2 - 2i]^2$

$= [1 - 1 - 2i]^2$ $[i^2 = -1]$

$= (-2i)^2$

$= 4(-1)$

$= -4$

Hence, $(1 - i)^4 = -4 + 0i$

Question 9

$(1/3 + 3i)^3$

Answer:

$\left(\frac{1}{3}+\ 3i\right)^3\ =\ \left(\frac{1}{3}\right)^3+(3i)^3+\ \left(\frac{1}{3}\right)\ (3i)\ \left(\frac{1}{3}+3i\right)$

$=\frac{1}{27}\ 27i^3\ +\ 3i\ \left(\frac{1}{3}+\ 3i\right)$

$=\frac{1}{27}\ 27\ (-i)+\ i+9i^2$ $[i^3\ =\ -i]$

$=\frac{1}{27}\ 27i+i-9$ $[i^2\ =\ -1]$

$=\left(\frac{1}{27}-\ 9\right)+\ i\ (-27+1)$

$= \frac{-242}{27} - 26i$

Hence, $(1/3 + 3i)^3 = - 242/27 - 26i$

Question 10

$(-2 - 1/3i)^3$

Answer:

$\left(-2 - \frac{1}{3}i\right)^3 = (-1)^3 \left(2 + \frac{1}{3}i\right)^3$

$= -\left[2^3 + \left(\frac{i}{3}\right)^3 + 3(2)\left(\frac{i}{3}\right)\left(2 + \frac{i}{3}\right)\right]$

$= -\left[8 + \frac{i^2}{27} + 2i\left(2 + \frac{i}{3}\right)\right]$

$= -\left[8 - \frac{i}{27} + 4i + \frac{2i^3}{3}\right] \qquad [i^3 = -i]$

$= -\left[8 - \frac{i}{27} + 4i - \frac{2}{3}\right] \qquad [i^2 = -i]$

$= -\left[\frac{22}{3} + \frac{107i}{27}\right]$

$= -\frac{22}{3} - \frac{107}{27}i$

Hence

$(-2 - 1/3i)^3 = 22/3 - 107/27i$

Question 11

Find the multiplicative inverse of each of the complex numbers given in the Exercises 11 to 13.

$4 - 3i$

Answer:

Let's consider $z = 4 - 3i$

Then,

$= 4 + 3i$ and

$|z|^2 = 4^2 + (-3)^2 = 16 + 9 = 25$

$|z|2 = 4^2 + (-3)^2 = 16 + 9 = 25$

Thus, the multiplicative inverse of $4 - 3i$ is given by z^{-1}

$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{4+3i}{25} = \frac{4}{25} + \frac{3}{25}i$

Question 12

$\sqrt{5} + 3i$

Answer:

Let's consider z = $\sqrt{5} + 3i$

$|z|^2 = (\sqrt{5})^2 + 3^2 = 5 + 9 = 14$

Thus, the multiplicative inverse of $\sqrt{5} + 3i$ is given by z^{-1}

$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{\sqrt{5} - 3i}{14} = \frac{\sqrt{5}}{14} - \frac{3i}{14}$

Question 13

– i

Answer:

Let's consider $z = -i$

Then, $\bar{z} = i$ and

$|z|^2 = 12 = 1$

Thus, the multiplicative inverse of $-i$ is given by z^{-1}

$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{i}{1}\ i$

Question 14

Express the following expression in the form of a + ib:

$$\frac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-i\sqrt{2})}$$

Answer:

$$\frac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-i\sqrt{2})}$$

$= \frac{(3)^2 - -(i\sqrt{5})^2}{\sqrt{3}+\sqrt{2}i-\sqrt{3}+\sqrt{2}i}$ $[(a+b)(a-b) = a^2 - b^2]$

$= \frac{9-5i^2}{2\sqrt{2}i}$

$= \frac{9-5(-1)}{2\sqrt{2}i}$ $[i^2 = -1]$

$= \frac{9+5}{2\sqrt{2}i} \times \frac{i}{i}$

$= \frac{14i}{2\sqrt{2}i^2}$

$= \frac{14i}{2\sqrt{2}(-1)}$

$= \frac{-7i}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{-7\sqrt{2}i}{2}$

Hence

$\frac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-i\sqrt{2})} = 0 + \frac{-7\sqrt{2}i}{2}$

Exercise 5.2

Question 1

Find the modulus and the arguments of each of the complex number in exercises 1 to 2
$z = -1 - i\sqrt{3}$

Answer:

Given,
$z = -1 - i\sqrt{3}$
Let $r\cos\theta = -1$ and $r \sin\theta = -\sqrt{3}$
On squaring and adding we get
$(r \cos\theta)^2 + (r \sin\theta)^2 = (-1)^2 + (-\sqrt{3})^2$
$r^2 (\cos^2\theta + \sin^2\theta) = 1 + 3$
$r^2 = 4$ $[\cos^2\theta + \sin^2\theta = 1]$
$r = \sqrt{4} = 2$ [conventionally, $r > 0$]
Thus modulus = 2
So we have
$2\cos\theta = -1$ and $2\sin\theta = -\sqrt{3}$
$\cos\theta = \frac{-1}{2}$ and $\sin\theta = \frac{-\sqrt{3}}{2}$
As the values of both $\sin\theta$ and $\cos\theta$ are negative, θ lies in III Quadrant,
Argument = $-\left(\pi - \frac{\pi}{3}\right) = \frac{-2\pi}{3}$
Therefore the modulus and argument of the complex $-1\sqrt{3}i$ are 2 and $\frac{-2\pi}{3}$ respectively

Question 2

$z = -\sqrt{3} + i$

Answer:

Given,
$z = -\sqrt{3} + i$
Let $r\cos\theta = -\sqrt{3}$ and $r \sin\theta = 1$
On squaring and adding we get
$r^2 \cos^2\theta + r^2 \sin^2\theta = (-\sqrt{3})^2 + 1^2$
$r^2 = 3 + 1 = 4$ $[\cos^2\theta + \sin^2\theta = 1]$

$r = \sqrt{4} = 2$ [conventionally, $r > 0$]
Thus modulus = 2
So we have
$2\cos\theta = -\sqrt{3}$ and $2\sin\theta = 1$
$\cos\theta = \frac{-\sqrt{3}}{2}$ and $\sin\theta = \frac{1}{2}$
$\therefore = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$
Therefore the modulus and argument of the complex $-\sqrt{3} + i$ are 2 and $\frac{5\pi}{6}$ respectively

Question 3

Convert each of the complex numbers given in exercises 3 to 8 in the polar form:
1 – i

Answer:

Given complex number
1 – i
Let $r\cos\theta = 1$ and $r\sin\theta = -1$
On squaring and adding we get
$r^2 \cos^2\theta + r^2 \sin^2\theta = 1^2 + (-1)^2$
$r^2 (\cos^2\theta + \sin^2\theta) = 1 + 1$
$r^2 = 2$
$r = \sqrt{2}$ = Modulus [conventionally, $r > 0$]
So
$\sqrt{2}\cos\theta = 1$ and $\sqrt{2}\sin\theta = -1$
$\cos\theta = \frac{1}{\sqrt{2}}$ and $\sin\theta = -\frac{1}{\sqrt{2}}$
$\therefore \theta = \frac{\pi}{4}$ [As θ lies in the IV quadrant]
So,
$1 - i = r\cos\theta + i\, r\sin\theta = \sqrt{2}\cos\left(-\frac{\pi}{4}\right) + \sqrt{2}\sin\left(-\frac{\pi}{4}\right)$
$= \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$
Hence this is the required polar form.

Question 4

$-1 + i$

Answer:

Given complex number
-1 + i

Let $r\cos\theta = -1$ and $r \sin\theta = 1$
On squaring and adding we get
$r^2 cos^2\theta + r^2 sin^2\theta = (-1)^2 + 1^2$
$r^2 (cos^2\theta + sin^2\theta) = 1 + 1$
$r^2 = 2$
$r = \sqrt{2} =$ [conventionally, $r > 0$]
So
$\sqrt{2}\cos\theta = -1$ and $\sqrt{2}\sin\theta = -1$
$\cos\theta = -\frac{1}{\sqrt{2}}$ and $\sin\theta = \frac{1}{\sqrt{2}}$
$\therefore \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ [As θ lies in the II quadrant]
$- + 1 + i = r\cos\theta + i\, r\sin\theta = \sqrt{2}\cos\frac{3\pi}{4} + i\sqrt{2}\sin\frac{3\pi}{4}$
$= \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$
This is the required polar from.

Question 5

$-1 + -i$

Answer:

Given complex number,
-1 – i
Let $r\cos\theta = -1$ and $r \sin\theta = -1$
On squaring and adding we get
$r^2 cos^2\theta + r^2 sin^2\theta = (-1)^2 + (-1)^2$
$r^2 (cos^2\theta + sin^2\theta) = 1 + 1$
$r^2 = 2$
$r = \sqrt{2} =$ [conventionally, $r > 0$]
So
$\sqrt{2}\cos\theta = -1$ and $\sqrt{2}\sin\theta = -1$
$\cos\theta = -\frac{1}{\sqrt{2}}$ and $\sin\theta = \frac{1}{\sqrt{2}}$
$\therefore \theta = \left(\pi - \frac{\pi}{4}\right) = -\frac{3\pi}{4}$ [As θ lies in the II quadrant]
Hence it can be written as
$- 1 + i = r\cos\theta + i\, r\sin\theta = \sqrt{2}\cos\frac{-3\pi}{4} + i\sqrt{2}\sin\frac{-3\pi}{4}$
$= \sqrt{2}\left(\cos\frac{-3\pi}{4} + i\sin\frac{-3\pi}{4}\right)$
This is the required polar from.

Question 6

-3

Answer:

Given complex number

-3

Let $r\cos\theta = -3$ and $r\ \sin\theta = 0$

On squaring and adding we get

$r^2\ cos^2\theta + r^2\ sin^2\theta = (-3)^2$

$r^2\ (cos^2\theta + sin^2\theta) = 9$

$r^2 = 9$

$r = \sqrt{9} = 3$ [conventionally, $r > 0$]

so

$3\cos\theta = -3$ and $3\sin\theta = 0$

$\Rightarrow \cos\theta = -1$ and $\sin\theta = 0$

$\therefore \theta = \pi$

Hence it can be written as

$-3 = r\cos\theta + i\ r\ \sin\theta = 3\cos\pi + i3\sin\pi = 3\ (\cos\pi + i\sin\pi)$

This is the required polar from

Question 7

$3 + i$

Answer:

Given complex number

$\sqrt{3} + i$

Let $r\cos\theta = \sqrt{3}$ and $r\ \sin\theta = 1$

On squaring and adding we get

$r^2\ cos^2\theta + r^2\ sin^2\theta = (\sqrt{3})^2 + 1^2$

$r^2\ (cos^2\theta + sin^2\theta) = 3 + 1$

$r^2 = 4$

$r = \sqrt{4} = 2$ [conventionally, $r > 0$]

so

$2\cos\theta = \sqrt{3}$ and $2\sin\theta = 1$

$\Rightarrow \cos\theta = \frac{\sqrt{3}}{2}$ and $\sin\theta = \frac{1}{2}$

$\therefore \theta = \frac{\pi}{6}$ [As θ lies in the I quadrant]

Hence it can be written as

$\sqrt{3} + i = r\cos\theta + i\ r\ sin\theta = 2\cos\frac{\pi}{6} + i\ 2\sin\frac{\pi}{6}$

$= 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

This is the required polar from

Question 8

i

Answer:

Given complex number, *i*

Let $r\cos\theta = 0$ and $r\sin\theta = 1$

On squaring and adding we get

$r^2\cos^2\theta + r^2\sin^2\theta = 0^2 + 1^2$

$r^2(\cos^2\theta + \sin^2\theta) = 1$

$r^2 = 1$

$r = \sqrt{1} = 1$ [conventionally, $r > 0$]

so

$\cos\theta = 0$ and $\sin\theta = 1$

$\therefore \theta = \frac{\pi}{2}$

Hence it can be written as

$i = r\cos\theta + i\, r\sin\theta = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

This is the required polar from

Exercise 5.3

Question 1

Solve each of the following equations:

$x^2 + 3 = 0$

Answer:

Given quadratic equation,

$x^2 + 3 = 0$

On comparing it with $ax^2 + bx + c = 0$, we have

$a = 1$, $b = 0$, and $c = 3$

So, the discriminated of the given equation will be

$D = b^2 - 4ac = 0^2 - 4 \times 1 \times 3 = -12$

Hence, the required solutions are:

$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{\pm \sqrt{-12}}{2 \times 1} = \frac{\pm \sqrt{12}i}{2}$ $[\sqrt{-1} = i]$

$\therefore x = \frac{\pm 2\sqrt{3}i}{2} = \pm\sqrt{3}i$

Question 2

$2x^2 + x + 1 = 0$

Answer:

Given quadratic equation,
$2x^2 + x + 1 = 0$
On comparing it with $ax^2 + bx + c = 0$, we have
a = 2, b = 1, and c = 1
So, the discriminate of the given equation will be
$D = b^2 - 4ac = 1^2 - 4 \times 2 \times 1 = 1 - 8 = -7$
Hence, the required solutions are:

$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-7}}{2 \times 2} = \frac{-1 \pm \sqrt{7}i}{4}$ $[\sqrt{-1} = i]$

Question 3

$x^2 + 3x + 9 = 0$

Answer:
Given quadratic equation,
$x^2 + 3x + 9 = 0$
On comparing it with $ax^2 + bx + c = 0$, we have
a = 1, b = 3, and c = 9
So, the discriminate of the given equation will be
D = b2 – 4ac = 32 – 4 × 1 × 9 = 9 – 36 = –27
Hence, the required solutions are:

$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-3 \pm \sqrt{-27}}{2\,(1)} = \frac{-3 \pm 3\sqrt{-3}}{2} = \frac{-3 \pm 3\sqrt{3}i}{2}$ $[\sqrt{-1} = i]$

Question 4

$-x^2 + x - 2 = 0$

Answer:

Given quadratic equation,
$-x^2 + x - 2 = 0$

On comparing it with $ax^2 + bx + c = 0$, we have
a = –1, b = 1, and c = –2
So, the discriminate of the given equation will be
$D = b^2 - 4ac = 1^2 - 4 \times (-1) \times (-2) = 1 - 8 = -7$
Hence, the required solutions are:

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-7}}{2 \times (1)} = \frac{-1 \pm \sqrt{7}i}{-2} \qquad [\sqrt{-1} = i]$$

Question 5

$x^2 + 3x + 5 = 0$

Answer:

Given quadratic equation,
$x^2 + 3x + 5 = 0$
On comparing it with ax2 + bx + c = 0, we have
a = 1, b = 3, and c = 5
So, the discriminated of the given equation will be
$D = b^2 - 4ac = 3^2 - 4 \times 1 \times 5 = 9 - 20 = -11$
Hence, the required solutions are:

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-3 \pm \sqrt{-7}}{2 \times 1} = \frac{-3 \pm \sqrt{11}i}{-2} \qquad [\sqrt{-1} = i]$$

Question 6

$x^2 - x + 2 = 0$

Answer:

Given quadratic equation,
$x^2 - x + 2 = 0$
On comparing it with ax2 + bx + c = 0, we have
a = 1, b = –1, and c = 2
So, the discriminate of the given equation is
$D = b^2 - 4ac = (-1)^2 - 4 \times 1 \times 2 = 1 - 8 = -7$
Hence, the required solutions are:

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-1) \pm \sqrt{-7}}{2 \times 1} = \frac{1 \pm \sqrt{7}i}{2} \qquad [\sqrt{-1} = i]$$

Question 7

$\sqrt{2}x^2 + x + \sqrt{2} = 0$

Answer:

Given quadratic equation,
$\sqrt{2}x^2 + x + \sqrt{2} = 0$
On comparing it with ax2 + bx + c = 0, we have
$a = \sqrt{2}$, b = 1, and $c = \sqrt{2}$
So, the discriminated of the given equation is
$D = b^2 – 4ac = (1)^2 – 4 \times \sqrt{2} \times \sqrt{2} = 1 – 8 = –7$
Hence, the required solutions are:

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-7}}{2 \times \sqrt{2}} = \frac{1 \pm \sqrt{7}i}{2\sqrt{2}} \qquad [\sqrt{-1} = i]$$

Question 8

$\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$

Answer:

Given quadratic equation,
$\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$
On comparing it with ax2 + bx + c = 0, we have
$a = \sqrt{3}$, $b = -\sqrt{2}$, and $c = 3\sqrt{3}$
So, the discriminate of the given equation is
$D = b^2 – 4ac = (-\sqrt{2})^2 – 4 \times \sqrt{3} \times 3\sqrt{3} = 2 – 36 = –34$
Hence, the required solutions are:

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-)\sqrt{2}) \pm 34}{2 \times \sqrt{3}} = \frac{\sqrt{2} \pm \sqrt{34i}}{2\sqrt{3}} \qquad [\sqrt{-1} = i]$$

Question 9

$x^2 + x + 1/\sqrt{2} = 0$

Answer:

Given quadratic equation,
$x^2 + x + 1/\sqrt{2} = 0$
It can be rewritten as,
$\sqrt{2}x^2 + \sqrt{2}x + 1 = 0$
On comparing it with ax2 + bx + c = 0, we have
$a = \sqrt{2}$, $b = \sqrt{2}$, and c = 1
So, the discriminate of the given equation is
$D = b^2 – 4ac = (\sqrt{2})^2 – 4 \times \sqrt{2} \times 1 = 2 – 4\sqrt{2} = 2(1 – 2\sqrt{2})$
Hence, the required solutions are:

$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-\sqrt{2} \pm \sqrt{2-4\sqrt{2}}}{2\times \sqrt{2}} = \frac{-\sqrt{2}\pm \sqrt{2(1-2\sqrt{2})}}{2\sqrt{2}}$ $[\sqrt{-1} = i]$

$= \left(\frac{-\sqrt{2}\pm \sqrt{2}\left(\sqrt{2\sqrt{2}-1}\right)i}{2\sqrt{2}}\right)$ $[\sqrt{-1} = i]$

$= \frac{-1 \pm \left(\sqrt{2\sqrt{2}-1}\right)i}{2}$

Question 10

$x^2 + x/\sqrt{2} + 1 = 0$

Answer:

Given quadratic equation,
$x^2 + x/\sqrt{2} + 1 = 0$
It can be rewritten as,
$\sqrt{2}x^2 + x + \sqrt{2} = 0$
On comparing it with $ax^2 + bx + c = 0$, we have
$a = \sqrt{2}$, $b = 1$, and $c = \sqrt{2}$
So, the discriminate of the given equation is
$D = b^2 - 4ac = (1)^2 - 4 \times \sqrt{2} \times \sqrt{2} = 1 - 8 = -7$
Hence, the required solutions are:

$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-7}}{2\sqrt{2}} = \frac{-1\pm \sqrt{7}\,i}{2\sqrt{2}}$ $[\sqrt{-1} = i]$

Miscellaneous Exercise

Question 1

Evaluate: $\left[i^{18} + \left(\frac{1}{i}\right)^{25}\right]^3$

Answer:

$\left[i^{18} + \left(\frac{1}{i}\right)^{25}\right]^3$

$= \left[i^{4\times 4+2} + \frac{1}{i^{4\times 6+1}}\right]^3$

$= \left[(i^4)^4\, i^2 + \frac{1}{(i4)^6\, i}\right]^3$

$= \left[i^3 + \frac{1}{i}\right]^3$ $[i^4 = 1]$

= $\left[-1+\frac{1}{i}\times\frac{i}{i}\right]^3$ $[i^2 = 1]$

= $\left[-1+\frac{i}{i^2}\right]^3$

= $[-1-i]^3$

= $(-1)^3\ [1+i]^3$

= -$[1^3 +\ i^3 +\ 3.1.i\ (1+i)]$

= - $[1+\ i^3 +\ 3i +\ 3i^2]$

= - $[1-i+3i-3]$

= - $[-2+2i]$

= 2 – 2i

Question 2

For any two complex numbers z_1 and z_2 prove that
Re $(z_1\ z_2)$ = Re z_1Re $z_2 -$ Im z_1 Im z_2

Answer:

Let's assume $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ as to complex number
Product of these complex numbers, $z_1\ z_2$
$z_1\ z_2 = (x_1 + iy_1)\ (x_2 + iy_2)$
$= x_1(x_2 + iy_2) + iy_1\ (x_2 + iy_2)$
$= x_1x_2 + ix_1\ y_2 + iy_1\ x_2 + i^2\ y_1y_2$
$= x_1x_2 + ix_1\ y_2 + iy_1\ x_2 - y_1y_2$ $[i^2 = -1]$
$= (x_1x_2 - y_1y_2) + i\ (x_2y_1 + y_1y_2)$
Now,
Re $(z_1\ z_2) = x_1x_2 - y_1y_2$
⇒Re $(z_1\ z_2)$ Re z_1Re $z_2 -$ Im z_1 Im z_2
Hence, proved

Question 3

Reduce$\left(\frac{1}{1-4i}-\frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$to the standard form

Answer:

$\left(\frac{1}{1-4i}-\frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)=\left[\frac{(1+i)-2\,(1-4i)}{(1-4i)(1+i)}\right]\left[\frac{3-4i}{5+i}\right]$

$=\left[\frac{1+i-2\;8\,i}{1+i-4i-4\,i^2}\right]\left[\frac{3-4i}{5+i}\right]=\left[\frac{-1+9i}{5-3i}\right]\left[\frac{3-4i}{5+i}\right]$

$=\left[\frac{-3+4i\;6\;27i-36i^2}{25+5i-15i-3i^2}\right]=\frac{33+31i}{28-10i}=\frac{33+31i}{2\,(14-5i)}$

$=\frac{(33+31i)}{2\,(14-5i)}\times\frac{(14+5i)}{(14+5i)}$ [on multiplying numerator and denominator by $(14+5i)$]

$=\frac{462+165i+434i+155i^2}{2\,[\,(14)^2-(5i)^2]}=\frac{307+599i}{2\,(196-25i^2)}$

$=\frac{307+599i}{2\,(221)}=\frac{307+599i}{442}=\frac{307}{442}+\frac{599i}{442}$

Question 4

If x = iy = $\sqrt{\frac{a-ib}{c-id}}$ prove that $(x^2+y^2)^2=\frac{a^2+b^2}{c^2+d^2}$

Answer:

Given,

$x-iy=\sqrt{\frac{a-ib}{c-id}}$

$=\sqrt{\frac{a-ib}{c-id}\times\frac{c+id}{c+id}}$ [on multiplying numerator and denominator by (c + id)]

$=\sqrt{\frac{(ac+bd)+i\,(ad-bc)}{c^2+d^2}}$

So,

$(x-iy)^2=\frac{(ac+bd)+i\,(ad-bc)}{c^2+d^2}$

$x^2\,y^2-2ixy=\frac{(ac+bd)+i\,(ad-bc)}{c^2+d^2}$

On comparing real and imaginary parts we get

$x^2-y^2=\frac{ac+bd}{c^2+d^2}\,,-2xy=\frac{ad-bc}{c^2+d^2}$ (1)

$(x^2+y^2)^2=(x^2-y^2)^2+4x^2\,y^2$

$=\left(\frac{ac+bd}{c^2+d^2}\right)^2+\left(\frac{ad+bc}{c^2+d^2}\right)^2$ [Using (1)]

$=\frac{a^2\,b^2+b^2\,d^2+2acbd+a^2\,d^2+b^2c^2-2adbc}{(c^2+d^2)^2}$

$=\frac{a^2\,b^2+b^2\,d^2+a^2\,d^2+b^2c^2}{(c^2+d^2)^2}$

$=\frac{a^2\,(c^2+d^2)+b^2\,(c^2+d^2)}{(c^2+d^2)^2}$

$=\frac{(c^2+d^2)\,(a^2+b^2)}{(c^2+d^2)^2}$

$=\frac{a^2+b^2}{c^2+d^2}$

Hence Proved

Question 5

Convert the following in the polar form:

(i) $\frac{1+7i}{(2-i)^2}$, **(ii)** $\frac{1+3i}{1+2i}$

Answer:

Here

(i) z = $\frac{1+7i}{(2-i)^2}$

$= \frac{1+7i}{(2-i)^2} = \frac{1+7i}{4+i^2\,4i} = \frac{1+7i}{4-1-4i}$

$= \frac{1+7i}{3-4i} \times \frac{3+4i}{3+4i} = \frac{3+4i+21i+28i^2}{3^2+4^2}$ [Multiplying by its conjugate in the numerator and denominator]

$= \frac{3+4i+21i+28i}{3^2+4^2} = \frac{-25+25i}{25}$

= - 1 + i

Let $r\cos\theta = -1$ and $r\ \sin\theta = 1$

On squaring and adding we get

$r^2\cos^2\theta + r^2\sin^2\theta = 1+1$

$r^2(\cos^2\theta + \sin^2\theta) = 2$

$r^2 = 1$ [$\cos^2\theta + \sin^2\theta = 1$]

$r = \sqrt{2}$ [conventionally, $r > 0$]

so

$\sqrt{2}\,\text{Cos}\,\theta = -1$ *and* $\sqrt{2}\sin\theta = 1$

$\Rightarrow \cos\theta = \frac{-1}{\sqrt{2}}$ *and* $\sin\theta = \frac{1}{\sqrt{2}}$

$\therefore \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ [As θ lies in II quadrant]

Expressing as, $z = r\cos\theta + i\,r\sin\theta$

$= \sqrt{2}\,\text{Cos}\,\frac{3\pi}{4} + i\sqrt{2}\sin\frac{3\pi}{4} = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

This is the required polar from

(ii) Let, $z = \frac{1+3i}{1+2i}$

$= \frac{1+3i}{1+2i} \times \frac{1+2i}{1+2i}$

$= \frac{1+2i+3i-6}{1+4}$

$= \frac{-5+5i}{5} = -1+i$

Now,

Let $r\cos\theta = -1$ and $r\ \sin\theta = 1$

On squaring and adding we get

$r^2(\cos^2\theta + r^2\sin^2\theta) = 1+1$

$r^2\ (\cos^2\theta + \sin^2\theta) = 2$

$r^2 = 1$ $[\cos^2\theta + \sin^2\theta = 1]$

$\Rightarrow r = \sqrt{2}$ [conventionally, $r > 0$]

so

$\therefore\ \sqrt{2}\operatorname{Cos}\theta = -1$ *and* $\sqrt{2}\sin\theta = 1$

$\cos\theta = \frac{-1}{\sqrt{2}}$ *and* $\sin\theta = \frac{1}{\sqrt{2}}$

$\therefore \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ [As θ lies in II quadrant]

Expressing as, $z = r\cos\theta + i\, r\sin\theta$

$Z = \sqrt{2}\cos\frac{3\pi}{4} + i\sqrt{2}\sin\frac{3\pi}{4} = \sqrt{2}\ \left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

This is the required polar from.

Question 6

Solve each of the equation in Exercises 6 to 9.

$3x^2 – 4x + 20/3 = 0$

Answer:

Given quadratic equation, $3x^2 – 4x + 20/3 = 0$

It can be re-written as: $9x^2 – 12x + 20 = 0$

On comparing it with $ax^2 + bx + c = 0$, we get

$a = 9$, $b = –12$, and $c = 20$

So, the discriminant of the given equation will be

$D = b^2 – 4ac = (–12)^2 – 4 \times 9 \times 20 = 144 – 720 = –576$

Hence, the required solutions are

$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-12) \pm \sqrt{-576}}{2\times 9} = \frac{12 \pm \sqrt{576}\,i}{18}$

$= \frac{12 \pm 24i}{18} = \frac{6\ 2(\pm 4i)}{18} = \frac{2 \pm 4i}{3} = \frac{2}{3} \pm \frac{4}{3}i$

Question 7

$x^2 – 2x + 3/2 = 0$

Answer:

Given quadratic equation, $x^2 – 2x + 3/2 = 0$

It can be re-written as $2x^2 – 4x + 3 = 0$

On comparing it with $ax^2 + bx + c = 0$, we get

$a = 2$, $b = –4$, and $c = 3$

So, the discriminant of the given equation will be

$D = b^2 – 4ac = (–4)^2 – 4 \times 2 \times 3 = 16 – 24 = –8$

Hence, the required solutions are

$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-4) \pm \sqrt{-8}}{2 \times 2} = \frac{4 \pm 2\sqrt{2}i}{18}$ $[\sqrt{-1} = i]$

$= \frac{2 \pm \sqrt{2}i}{2} = 1 \pm \frac{\sqrt{2}}{2} i$

Question 8

$27x^2 - 10x + 1 = 0$

Answer:

Given quadratic equation, $27x^2 - 10x + 1 = 0$
On comparing it with $ax^2 + bx + c = 0$, we get
a = 27, b = –10, and c = 1
So, the discriminant of the given equation will be
D = b^2 – 4ac = (–10)2 – 4 × 27 × 1 = 100 – 108 = –8
Hence, the required solutions are

$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-10) \pm \sqrt{-8}}{2 \times 27} = \frac{10 \pm 2\sqrt{2}i}{54}$

$= \frac{5 \pm \sqrt{2}\, i}{27} = \frac{5}{27} \pm \frac{\sqrt{2}}{27} i$

Question 9

$21x^2 - 28x + 10 = 0$

Answer:

Given quadratic equation, $21x^2 - 28x + 10 = 0$
On comparing it with $ax^2 + bx + c = 0$, we have
a = 21, b = –28, and c = 10
So, the discriminant of the given equation will be
D = b^2 – 4ac = (–28)2 – 4 × 21 × 10 = 784 – 840 = –56
Hence, the required solutions are:

$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-28) \pm \sqrt{-56}}{2 \times 21} = \frac{28 \pm \sqrt{56}i}{42}$

$= \frac{28 + 2\sqrt{4}\, i}{42} = \frac{28}{42} \pm \frac{2\sqrt{14}}{42} i \; \frac{2}{3} \pm \frac{\sqrt{14}}{21} i$

Question 10

If $z_1 = 2 - i, z_2 = 1 + i$, find $\left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + 1}\right|$

Answer:

Given $z_1 = 2 - i, z_2 = 1 + i$

So,

$\left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + 1}\right| = \left|\frac{(2-i) + (1+i) + 1}{(2-i) - (1+i) + 1}\right|$

$= \left|\frac{4}{2-2i}\right| = \left|\frac{4}{2\,(1-i)}\right|$

$= \left|\frac{2}{1-i} \times \frac{1+i}{1+i}\right| = \left|\frac{2\,(1+i)}{1^2 - i^2}\right|$

$= \left|\frac{2\,(1+i)}{1+1}\right| \quad [i^2 = -1]$

$= \left|\frac{2\,(1+i)}{2}\right|$

$= |1 + \mathrm{i}| = \sqrt{1^2 + 1^2} = \sqrt{2}$

Hence the value of $\left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + 1}\right|$ is $\sqrt{2}$.

Question 11

If $\alpha + ib = \frac{(x+i)^2}{2x^2\ i}$, prove that $\alpha^2 + b^2 = \frac{(x^2 + 1)^2}{(2x^2 + 1)^2}$

Answer:

$a + ib = \frac{(x+i)^2}{2x^2\ i}$

$= \frac{x^2 + i^2 + 2xi}{2x^2 + 1}$

$= \frac{x^2 - 1 + i2x}{2x^2 + 1}$

$= \frac{x^2 - 1}{2\,x^2 + 1} + i\left(\frac{2x}{2x^2 + 1}\right)$

Comparing the real imaginary parts we have

$a = \frac{x^2 - 1}{2\,x^2 + 1}$ and $b = \frac{2x}{2x^2 + 1}$

$\therefore a^2 + b^2 = \left(\frac{x^2 - 1}{2x^2 + 1}\right)^2 + \left(\frac{2x}{2x^2 + 1}\right)^2$

$= \frac{x^2 + 1 - 2x^2 + 4\,x^2}{(2x+1)^2}$

$= \frac{x^4 + 1 + 2\,x^2}{(2x^2 + 1)^2}$

$= \frac{(x^2 + 1)^2}{(2x^2 | 1)^2}$

Hence proved

$a^2 + b^2 = \frac{(x^2 + 1)^2}{(2x^2 + 1)^2}$

Question 12

Let $z_1 = 2 - i, z_1 = -2 + i$ find

(i) Re$\left(\frac{z_1 z_2}{z_1}\right)$ (ii) Im $\left(\frac{1}{z_1 \bar{z}_1}\right)$

Answer:

Given

$z_1 = 2 - i, z_1 = -2 + i$

(i) $z_1 z_2 = (2 - i)(-2 + i) = -4 + 2i + 2i - i^2 = -3 + 4i\ \bar{z}_1 = 2i$

$\therefore \frac{z_1 z_2}{z_1} = \frac{-3+4i}{2+i}$

On multiplying numerator and denominator by $(2 - i)$. We get

$\frac{z_1 z_2}{z_1} = \frac{(-3+i)(2-i)}{(2+i)(2+i)} = \frac{-6+3i+8i-4i^2}{2^2+1^2} = \frac{-6+3i+8i-4(-1)}{2^2+1^2}$

$= \frac{-2+||i}{2} = \frac{-2}{5} + \frac{11}{2}i$

Re$\left(\frac{z_1 z_2}{z_1}\right) = \frac{-2}{5}$

(ii) $\frac{1}{z_1 \bar{z}_1} = \frac{1}{(2-i)(2+i)} = \frac{1}{(2)^2+(1)^2} = \frac{1}{5}$

On comparing the imaginary part we get

Im $\left(\frac{1}{z_1 \bar{z}_1}\right) = 0$

Question 13

Find the modulus and argument of the complex number $\frac{1+2i}{1-3i}$

Answer:

Let z = $\frac{1+2i}{1-3i}$ then

$z = \frac{1+2i}{1-3i} \times \frac{1+2i}{1-3i} = \frac{1+3i+2i+6i^2}{i^2+3^2} = \frac{1+5i+6(-1)}{1+9}$

$= \frac{-5+5i}{10} = \frac{-5}{10} + \frac{5i}{10} = \frac{-1}{2} + \frac{1}{2}i$

Let $r\cos\theta + r\sin\theta = 1$

So

$r\cos\theta = \frac{-1}{2} \sin\theta = \frac{1}{2}$

On squaring and adding we get

$r^2(\cos^2\theta + r^2\sin^2\theta) = 1 + 1$

$r^2(\cos^2\theta + \sin^2\theta) = \left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2$

$r^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ [conventionally, $r > 0$]

$r = \frac{1}{\sqrt{2}}$

Now

$\therefore \frac{1}{\sqrt{2}}$ Cos θ $= \frac{-1}{2}$ and $\frac{1}{\sqrt{2}}$sin θ $= \frac{1}{2}$

⇨ $\cos\theta = \frac{-1}{\sqrt{2}}$ and $\sin\theta = \frac{1}{\sqrt{2}}$

$\therefore \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ [As θ lies in II quadrant]

Question 14

Find the real number x and y $(x - iy)(3 + 5i)$ is the conjugate of – 6 – 4i

Answer:

Let`s assume z =$(x - iy)(3 + 5i)$

z = $3x + 5xi - 3iy - 5yi^2 = 3x + 5xi - 3yi + 5y = (3x + 5y) + i(5x - 3y) \therefore \bar{z}$ = (3x + 5y) − i (5x − 3y)

Also given $\bar{z} = -6 - 24i$

And

(3x + 5y) − i (5x − 3y) $= -6 - 24i$

On equating real and imaginary parts we have

3x + 5y = −6 … … … … (i)

5x − 3y = 24 … … … … (ii)

Performing (i) x + (ii) x 5, we get

(9x + 15y) + (25x – 15y) = - 18 + 120

34x = 102

x = 102/34 = 3

Putting the value of x in equation (i), we get

3 (3) + 5y = -6

5y = - 6 – 9 = - 5

y = - 3

Therefore the value of x and y are 3 and – 3 respectively

Question 15

Find the modulus of $\frac{1+i}{1-i} - \frac{1-i}{1+i}$

Answer:

$$\frac{1+i}{1-i} - \frac{1-i}{1+i} = \frac{(1+i)^2 - (1-i)^2}{(1-i)(1+i)}$$

$$= \frac{1+i^2+2i-1-i^2+2i}{1^2+1^2}$$

$$= \frac{4i}{2} = 2i$$

$\therefore \left|\frac{1+i}{1-i} - \frac{1-i}{1+i}\right| = |2i| = \sqrt{2^2} = 2$

Question 16

If $(x+iy)^3\ u + iv$, then show that $\frac{u}{x} + \frac{v}{y} = 4\ (x^2 - y^2)$

Answer:

$(x+iy)^3\ u + iv,$
$x^3 + (iy)^3 + 3.x.iy\ (x+iy) = u + iv$
$x^3 + i^3\ y^3 + 3x^2\ yi - 3xy^2 i^2 = u + iv$
$x^3 - iy^3 + 3x^2\ yi - 3xy^2 = u + iv$
$(x^3 - 3xy^2) + i\ (3x^2\ y - y^3) = u + iv$
On equating real and imaginary parts we get
$u = x^3 - 3xy^2\ , v = 3x^2\ y - y^3$
$\frac{u}{x} + \frac{v}{y} = \frac{x^3 - 3xy^2}{x} + \frac{3x^2\ y - y^3}{y}$
$= \frac{x\ (x^2 - 3y^2)}{x} + \frac{y\ (3x^2 - y^2)}{y}$
$= x^2 - 3y^2 + 3x^2 - y^2$
$= 4x^2 - 4y^2$

$= 4\ (x^2 - y^2)$
$\therefore \frac{u}{x} + \frac{v}{y} = 4\ (x^2 - y^2)$

Question 17

If α and β are different cmplex numbers with $|\beta| = 1$, then find $\left|\frac{\beta - \alpha}{1 - \overline{\alpha}\beta}\right|$

Answer:

Let α = a + ib and $\beta = x + iy$
Given $|\beta| = 1$
So $\sqrt{x^2 + y^2} = 1$
$\Rightarrow x^2 + y^2 = 1$ ...... (i)
$\left|\frac{\beta - \alpha}{1 - \overline{\alpha}\beta}\right| = \left|\frac{(x+iy) - (a+ib)}{1 - (a - ib)(x+iy)}\right|$
$= \left|\frac{(x-a) - (a - ib)}{1 - (ax + aiy - ibx + by)}\right|$

$= \left|\frac{(x-a) - i\ (y-b)}{1 - (ax - by) + i\ (bx - ay)}\right|$ $\left[\left|\frac{z_1}{z_2}\right| = \left|\frac{z_1}{z_2}\right|\right]$

$$= \frac{\sqrt{(x-a)^2 + (y-b)^2}}{\sqrt{((1-ax-by)^2 + (bx-ay)^2}}$$

$$= \frac{\sqrt{x^2 + a^2 - 2ax + y^2 + b^2 - 2by}}{\sqrt{1 + a^2 x^2 + b^2 y^2 - 2ax + 2abxy - 2by + b^2 x^2 + a^2 y^2 - 2abxy}}$$

$$= \frac{\sqrt{(x^2 + y^2) + a^2 + b^2 - 2ax - 2by}}{\sqrt{1 + a^2 (x^2 + y^2) + b^2 (y^2 + x^2) - 2ax - 2by}}$$

$$= \frac{\sqrt{1 + a^2 + b^2 - 2ax - 2by}}{\sqrt{1 + a^2 + b^2 - 2ax - 2by}} \qquad \text{[Using (1)]}$$

= 1

$\therefore \left|\frac{\beta-\alpha}{1-\overline{\alpha}\beta}\right| = 1$

Question 18

Find the number of non –zero integral solution of the equation $|1-i|^x = 2^x$

Answer:

$|1-i|^x = 2^x$

$\left(\sqrt{1^2 + (-1)^2}\right)^x = 2^x$

$\left(\sqrt{2}\right)^x = 2^x$

$2^{\frac{x}{2}} = 2^x$

$\frac{x}{2} = x$

2x – x = 0

x = 0

Question 19

If (a+ ib) (c + id) (e+ if) (g + ih) = A + iB

$\therefore \left|(a^2 + b^2)(c^2 + d^2)(e^2 + f^2)(g^2 + h^2)\right| = A^2 + B^2$

Answer:

(a+ ib) (c + id) (e+ if) (g + ih) = A + iB

$\therefore |(a + ib)\,(c + id)\,(e + if)\,(g + ih)| = |A + iB|$

$\Rightarrow |(a + ib)| \times |(c + id)| \times |(e + if)| \times |(g + ih)| = |A + iB| \qquad [|z_1 z_2| = |z_1|\,|z_2|]$

$\sqrt{a^2 + b^2} \times \sqrt{c^2 + d^2} \times \sqrt{e^2 + f^2} \times \sqrt{g^2 + h^2} = \sqrt{A^2 + B^2}$

On squaring both side we get

$(a^2 + b^2)(c^2 + d^2)(e^2 + f^2)(g^2 + h^2)A^2 + B^2$

Question 20

If $\left(\frac{1+i}{1-i}\right)^m$ = 1 then find the last positive integral of m,

Answer:

$\left(\frac{1+i}{1-i}\right)^m = 1$

$\left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^m = 1$

$\left(\frac{(1+i)^2}{1^2 + 1^2}\right)^m = 1$

$\left(\frac{1^2 + i^2 + 2i}{2}\right)^m = 1$

$\left(\frac{2i}{2}\right)^m = 1$

$i^m = 1$

Hence, m = 4 k, where k is some integer

Thus the least positive integer is 1

Therefore, the least positive integral value of m is 4 (= 4 × 1)