Chapter 4
Principal of Mathematical Induction

Exercise 4.1

Question 1

Prove the following by using the principle of mathematical induction for all n ∈ N:
$1 + 3 + 3^2 + \ldots + 3^{n+1} = \frac{(3^n - 1)}{2}$

Answer:

We can write the given statement as
P (n): $1 + 3 + 3^2 + \ldots + 3^{n+1} = \frac{(3^n - 1)}{2}$
If n = 1 we get
P (n): $1 = \frac{(3^1 - 1)}{2}\ \frac{3-1}{2} = \frac{2}{2} = 1$
Which is true?
Consider P (k) be true for some positive integer k
$1 + 3 + 3^2 + \ldots + 3^{k+1} = \frac{(3^k - 1)}{2}$ ………. (1)
Now let us prove that P (K +1) is true
Here
$1 + 3 + 3^2 + \ldots + 3^{k-1} + 3^{(k+1)} = (1 + 3 + 3^2 + \ldots + 3^{k-1}) + 3^k$
By using equation (i)
$= \frac{(3^k - 1)}{2} + 3^k$
Taking LCM
$= \frac{(3^k - 1) + 2.3^K}{2}$
Taking the common terms out
$\frac{(1+2)3^k - 1}{2}$
We get
$= \frac{3.3^k - 1}{2}$
$= \frac{3^{k+1} - 1}{2}$
P (k + 1) is true whenever P (k) is rue
Therefore by the principal of mathematical induction, statement P(n) is true for all natural number i.e. n

Question 2

$$\mathbf{1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n\,(n+1)}{2}\right)^2}$$

Answer:

We can write the given statement as

P (n) $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n\,(n+1)}{2}\right)^2$

If n = 1 we get

P (1): $1^3 = 1 = \left(\frac{1\,(1+1)}{2}\right)^2 = \left(\frac{1.2}{2}\right)^2 = 1^2 = 1$

Which is true?

Consider P (k) be true for some positive integer k

$1^3 + 2^3 + 3^3 + \ldots + k^3 = \left(\frac{k(k+1)}{2}\right)^2$ ....(1)

Now let us prove that P (k + 1) is true

Here

$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = (1^3 + 2^3 + 3^3 + \ldots\ldots + k^3) + (k+1)^3$

By using equation (1)

$= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$

So we get

$\frac{k^2\,(k+1)^2}{4} + (k+1)^3$

Taking LCM

$\frac{k^2\,(k+1)^2 + 4\,(k+1)^3}{4}$

Taking the common terms out

$= \frac{(k+1)^2\,\{k^2 + 4\,(k+1)\}}{4}$

$= \frac{(k+1)^2 (k+1)^2}{4}$

= By expending using formula

$= \frac{(k+1)^2 (k+1+1)^2}{4}$

$= \left(\frac{(k+1)^2 (k+1+1)^2}{4}\right)^2$

P (k + 1) is true whenever P (k) is rue

Therefore by the principal of mathematical induction, statement P(n) is true for all natural number i.e. n

Question 3

$$\mathbf{1 + \frac{1}{(1+2)} + \frac{1}{(1+2+3)} + \cdots + \frac{1}{(1+2+3+ .n)} = \frac{2n}{(n+1)}}$$

Answer:

We can write the given statement as

$P(n): 1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \cdots + \frac{1}{1+2+3+.n} = \frac{2n}{n+1}$

If n = 1 we get

$P(1): 1 = \frac{2.1}{1+1} = \frac{2}{2} = 1$

Which is true?

Consider P (k) be true for some positive integer k

$1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \cdots + \frac{1}{1+2+3+\cdots+.k} = \frac{kn}{k+1}$ ……….(1)

Now let us prove that P (k + 1) is true

Here

$1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \cdots + \frac{1}{1+2+3+.k} + \frac{1}{1+2+3+\cdots.+k+(k+1)}$

$= \left(1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \cdots + \frac{1}{1+2+3+.k}\right) + \frac{1}{1+2+3+\cdots.+k+(k+1)}$

By using equation (i)

$= \frac{2k}{k+1} + \frac{1}{1+2+3+\cdots.+k+(k+1)}$

We know that

$1 + 2 + 3 + \ldots. + n = \frac{n(n+1)}{2}$

So we get

$= \frac{2k}{k+1} + \frac{1}{\left(\frac{(k+1)(k+1+1)}{2}\right)}$

It can be written as

$= \frac{2k}{(k+1)} + \frac{2}{(k+1)(k+2)}$

Taking the common term out

$= \frac{2k}{(k+1)} + \left(k + \frac{1}{k+2}\right)$

By taking LCM

$= \frac{2k}{k+1} + \left(\frac{k(k+2)+1}{k+2}\right)$

We get

$= \frac{2k}{(k+1)} \left(\frac{k^2 + 2k+1}{k+2}\right)$

$= \frac{2k(k+1)^2}{(k+1)(k+2)}$

$= \frac{2(k+1)}{(k+2)}$

P (k + 1) is true whenever P (k) is true

Therefore by the principal of mathematical induction, statement P(n) is true for all natural number i.e. n

Question 4

$1.2.3. + 2.3.4 + \ldots.. + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$

Answer:

We can write the given statement as

P (n): 1.2.3. + 2.3.4 +…..+ n (n+1) (n+2) = $\frac{n\,(n+1)\,(n+2)\,(n+3)}{4}$

If n = 1 we get

P (1): 1.2.3. = 6 = $\frac{1\,(1+1)\,(1+2)\,(1+3)}{4} = \frac{1.2.3.4}{4} = 6$

Which is true?

Consider P (k) be true for some positive integer k

1.2.3. + 2.3.4 +…..+ k (k+1) (k+2) = $\frac{k\,(k+1)\,(k+2)\,(k+3)}{4}$ ……… (1)

Now let us prove that P (k + 1) is true

Here

1.2.3. + 2.3.4 +…..+ k (k+1) (k+2) + (k+3) = {1.2.3. + 2.3.4 + ⋯ . . + k (k + 1) (k + 2)} + (k + 1)(k + 2(k+3)

By using equation (i)

= $\frac{k\,(k+1)\,(k+2)\,(k+3)}{4} + (k+1)\,(k+2)(k+3)$

So we get

= $(k+1)(k+2)(k+3)\left(\frac{k}{4} + 1\right)$

It can be written as

= $\frac{(k+1)(k+2)(k+3)(k+4)}{4}$

By further simplification

= $\frac{(k+1)(k+1+1)(k+1+2)(k+1+3)}{4}$

P (k + 1) is true whenever P (k) is true

Therefore by the principal of mathematical induction, statement P(n) is true for all natural number i.e. n

Question 5

1.3 + 2.3²+3.3² +…. + n.3ⁿ = $\frac{(2n-1)3^{n-1}+3}{4}$

Answer:

We can write the given statement as

P (n): 1.3 + 2.3^2+3.3^2 + ….+ $n.3^n$ = $\frac{(2n-1)3^{n-1}+3}{4}$

If n = 1 we get

P (1): 1.3. = 3 = $\frac{(2n-1)3^{n-1}+3}{4} = \frac{3^2+3}{4} = \frac{12}{4} = 3$

Which is true?

Consider P (k) be true for some positive integer k

$1.3 + 2.3^2 + 3.3^2 + ... + k.3^k = \frac{(2k-1)3^{k-1} + 3}{4}$ …….. (1)

Now let us prove that p (k + 1) is true

Here

$1.3 + 2.3^2 + 3.3^2 + ...+ k.3^k + (k + 1)\ 3^{k+1} = = (1.3 + 2.3^2 + 3.3^2 + ...+ k.3^k) + (k + 1)3^{k+1}$

By using equation (1)

$= \frac{(2k-1)3^{k-1} + 3}{4} + (k + 1)3^{k+1}$

By taking LCM

$= \frac{(2k-1)3^{k+1} + 3 + 4\ (k+1)3^{k+1}}{4}$

Taking the common terms out

$= \frac{3^{k+1}\ \{2\ k-1+4\ (k+1\ )\}+ 3}{4}$

By further simplification

$= \frac{3^{k+1}\ \ \{6k+3\}+ 3}{4}$

Taking 3 as common

$= \frac{3^{k+1}\ 3\ \{2k+1\}+ 3}{4}$

$= \frac{3^{(k+1)+1}\ \{2\ k+1\}+ 3}{4}$

$= \frac{\{2\ (k+1)- 1\}3^{(k+1)+1}\ +3}{4}$

P (k + 1) is true whenever P (k) is true

Therefore by the principal of mathematical induction, statement P(n) is true for all natural number i.e. n

Question 6

$1.2 + 2.3 + 3.4 + ... + n.(n + 1) = \left[\frac{n\ (n+1)(n+2)}{3}\right]$

Answer:

We can write the given statement as

P (n): $1.2 + 2.3 + 3.4 + ... + n.(n + 1) = \left[\frac{n\ (n+1)(n+2)}{3}\right]$

If n = 1 we get

P (1): $1.2 = 2 = \frac{1\ (1+1)(1+2)}{3} = \frac{1.2.3}{3} = 2$

Which is true?

Consider P (k) be true for some positive integer k

$1.2 + 2.3 + 3.4 + ... + k.\ (k + 1) = \left[\frac{k\ (k+1)(k+2)}{3}\right]$ …….. (1)

Now let us prove that p (k + 1) is true

Here

1.2 + 2.3 + 3.4 +.... +k. (k+1) + (k+2) = [1.2 + 2.3 + 3.4 + ...+ k (k+1)] + (k+1). (k+2)
By using equation (1)
$= \frac{k\,(k+1)(k+2)}{3} + (k+1)(k+2)$
We can write it as
$= (k+1)(k+2)\left(\frac{k}{3} + 1\right)$
We get
$= \frac{(k+1).(k+2).(k+3).}{3}$
By further simplification
$= \frac{(k+1).(k+1+1).(k+1+2)}{3}$
P (k + 1) is true whenever P (k) is true
Therefore by the principal of mathematical induction, statement P(n) is true for all natural number i.e. n

Question 7

1.3 + 3.5 + 5.7 +...+ (2n – 1) (2n +1) = $\frac{n\,(4n^2 + 6n - 1)}{3}$

Answer:

We can write the given statement as
P (n): 1.3 + 3.5 + 5.7 +...+ (2n – 1) (2n +1) = $\frac{n\,(4n^2 + 6n - 1)}{3}$
If n = 1 we get
P (1): 1.3 = 3 = $\frac{1\,(41^2 + 6.1 - 1)}{3} = \frac{4+6-1}{3} + \frac{9}{3} = 3$
Which is true?
Consider P (k) be true for some positive integer k
1.3 + 3.5 + 5.7 +...+ (2k – 1) (2k +1) = $\frac{k\,(4k^2 + 6k - 1)}{3}$ .... ..(1)
Now let us prove that p (k + 1) is true
Here
(1.3 + 3.5 + 5.7 +...+ (2k – 1) (2k +1) + {2 (k + 1) – 2} {2 (k + 1) + }
By using equation (1)
$= \frac{k\,(4k^2 + 6k - 1)}{3}$ + (2k +2 - 1) (2k +2 + 1)
$= \frac{k\,(4k^2 + 6k - 1)}{3}$ + (2k +2 - 1) (2k +2 + 1)
On further calculation
$= \frac{k\,(4k^2 + 6k - 1)}{3}$ + (2k +1) (2k +3)
By multiplying the terms
$= \frac{k\,(4k^2 + 6k - 1)}{3} + (4k^2 + 8k + 3)$
Taking LCM

$= \frac{k\,(4k^2+6k-1)+3\,(4\,k^2+8\,k+3)}{3}$

By further simplification

$= \frac{4k^3+6k^2-k+12\,k^2+24k+9}{3}$

So we get

$= \frac{4k^3+18k^2+23k+9}{3}$

It can be written as

$= \frac{4k^3+14\,k^2+9k+4k^2+14\,k+9}{3}$

$= \frac{k\,(4k^2+14\,k+9\,)+(4k^2+14\,k+9\,)}{3}$

Taking the common terms out

$= \frac{(k+1)\,\{4(k^2+2k+1)+6\,(k+1)-1\}}{3}$

Using the formula

$= \frac{(k+1)\,\{4\,(k+1)^2+6\,(k+1)-1\}}{3}$

P (k + 1) is true whenever P (k) is true

Therefore by the principal of mathematical induction, statement P(n) is true for all natural number i.e. n

Question 8

$1.2 + 2.2^2 + 3.2^2 + \ldots + n.2^n = (n - 1)\, 2^{n+1} + 2$

Answer:

We can write the given statement as

P (n) : $1.2 + 2.2^2 + 3.2^2 + \ldots + n.2^n = (n - 1)\, 2^{n+1} + 2$

If n = 1 we get

P (1): $1.2 = 2 = (1\text{-}1)^{1+1} + 2 = 0 + 2 = 2$

Which is true?

Consider P (k) be true for some positive integer k

$1.2 + 2.2^2 + 3.2^2 + \ldots + k2^k = (k - 1)\, 2^{n+1} + 2$ …… (1)

Now let us prove that p (k + 1) is true

Here

$\{1.2 + 2.2^2 + 3.2^2 + \ldots + k.2^k\} = (k - 1)\, 2^{k+1}$

By using equation (1)

$(k - 1)2^{k+1} + 2 + (k - 1)\, 2^{k+1}$

Taking the common terms out

$= 2^{k+1}\{(k - 1)\, (k - 1)\} + 2$

So we get

$= 2^{k+1}.\, 2k + 2$

It can be written as

= k. $2^{(k+1)\ -1}$+2
= {(k – 1)-1} $2^{(k+1)}$ +2
P (k + 1) is true whenever P (k) is true
Therefore by the principal of mathematical induction, statement P (n) is true for all natural number i.e. n

Question 9

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ \dots.+\frac{1}{2^n}=1-\frac{1}{2^n}$$

Answer:

We can write the given statement as
P (n): $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ \dots.+\frac{1}{2^n}=1-\frac{1}{2^n}$
If n = 1 we get
P (1): $\frac{1}{2} = 1-\frac{1}{2^n} = \frac{1}{-2}$
Which is true?
Consider P (k) be true for some positive integer k
$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ \dots.+\frac{1}{2^k}=1-\frac{1}{2^k}$ …….(1)
Now let us prove that p (k + 1) is true
Here
$\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ \dots.+\frac{1}{2^k}\right)+\frac{1}{2^{k+1}}$
By using equation (i)
$=\left(1-\frac{1}{2^k}\right)+\frac{1}{2^{k+1}}$
We can write it as
$=1-\frac{1}{2^k}+\frac{1}{2.2^k}$
Taking the common terms out
$=1-\frac{1}{2^k}\left(1-\frac{1}{2}\right)$
So we get
$=1-\frac{1}{2^k}\left(\frac{1}{2}\right)$
It can be written as
$=\frac{1}{2^{k+1}}$
P (k + 1) is true whenever P (k) is true
Therefore by the principal of mathematical induction, statement P(n) is true for all natural number i.e. n

Question 10

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+.....+\frac{1}{(3n+1)(3n+2)}=\frac{n}{(6n+4)}$$

Answer:

We can write the given statement as

$P(n): \frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+.....+\frac{1}{(3n+1)(3n+2)}=\frac{n}{(6n+4)}$

If n = 1 we get

$P(1): = \frac{1}{2.5} = \frac{1}{10} = \frac{1}{6.1+4} = \frac{1}{10}$

Which is true?

Consider P (k) be true for some positive integer k

$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+.....+\frac{1}{(3k+2)(3k+2)}=\frac{k}{6k+4}$ .... (i)

Now let us prove that p (k + 1) is true

Here

$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+.....+\frac{1}{(3k-)(3k+2)}+\frac{1}{\{3(k+1)-1\}\{3(k+1)+2\}}$

By using equation (i)

$=\frac{k}{(6k+4)}+\frac{1}{(3k+3-1)(3k+3+2)}$

By simplification of terms

$=\frac{k}{(6k+4)}+\frac{1}{(3k+2)(3k+5)}$

Taking 2 as common

$=\frac{k}{2(3k+2)}+\frac{1}{(3k+2)(3k+5)}$

Taking the common terms out

$=\frac{k}{(3k+2)}\left(\frac{k}{2}+\frac{1}{3k+5}\right)$

Taking LCM

$=\frac{1}{(3k+2)}\left(\frac{k(3k+5)+2}{2(3k+5)}\right)$

By multiplication

$\frac{1}{(3k+2)}\left(\frac{3k^2+5k+2}{2(3k+5)}\right)$

Separating the terms

$\frac{1}{(3k+2)}\left(\frac{(3k+)(k+1)}{2(3k+5)}\right)$

By further calculation

$$=\frac{(k+1)}{6k+10}$$

So we get

$=\frac{(k+1)}{6(k+1)+4}$

P (k + 1) is true whenever P (k) is true

Therefore by the principal of mathematical induction, statement P(n) is true for all natural number i.e. n

Question 11

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots..+\frac{1}{n\,(n+1)(n+2)}=\frac{n\,(n+3)}{4\,(n+1)(n+2)}$$

Answer:

We can write the given statement as

$$P\,(n): \frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots..+\frac{1}{n\,(n+1)(n+2)}=\frac{n\,(n+3)}{4\,(n+1)(n+2)}$$

If n = 1 we get

$$P\,(1): = \frac{1}{1.2.3}=\frac{1.(1+3)}{4\,(1+1)(1+2)}=\frac{1.4}{4.2.3}=\frac{1}{1.2.3}$$

Which is true?

Consider P (k) be true for some positive integer k

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots..+\frac{1}{k\,(k+1)(k+2)}=\frac{k\,(k+3)}{4\,(k+1)(k+2)} \quad \ldots\ldots (1)$$

Now let us prove that p (k + 1) is true

Here

$$\left[\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\cdots..+\frac{1}{k\,(k+1)(k+2)}\right]+\frac{1}{(k+1)(k+2)(k+3)}$$

By using equation (1)

$$=\frac{k\,(k+3)}{4(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}$$

Taking out the common terms

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k\,(k+3)}{4}+\frac{1}{k+3}\right\}$$

Taking LCM

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k\,(k+3)^2+4}{4\,(k+3)}\right\}$$

Expanding using formula

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k\,(k^2+6k+9)+4}{4\,(k+3)}\right\}$$

By further calculation

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k^3\,(6k^2+9k+4}{4\,(k+3)}\right\}$$

We can write it as

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k^3+2k^2\,k+4k^2+8k+4}{4\,(k+3)}\right\}$$

Taking the common terms

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k\,(k^2+2k+1)+4\,(k^2+2k+1)}{4\,(k+3)}\right\}$$

We get

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k\,(k+1)^2+4\,(k+1)^2}{4\,(k+3)}\right\}$$

Here

$= \frac{(k+1)^2 (k+4)}{4 (k+1)(k+2)(k+3)}$

$= \frac{(k+1)\{(k+1)(k+3)\}}{4 \{(k+1)+ 1\}\{(k+1)+ 2\}}$

P (k + 1) is true whenever P (k) is true

Therefore by the principal of mathematical induction, statement P(n) is true for all natural number i.e. n

Question 12

$\alpha + \alpha r + \alpha r^2 + ... + \alpha r^{r-1} = \frac{\alpha (r^n - 1)}{r-1}$

Answer:

We can write the given statement as

P (n): $\alpha + \alpha r + \alpha r^2 + ... + \alpha r^{r-1} = \frac{\alpha (r^n - 1)}{r-1}$

If n = 1 we get

P (1): = a= $\frac{a (r^1 - 1)}{r-1}$ = α

Which is true?

Consider P (k) be true for some positive integer k

$\alpha + \alpha r + \alpha r^2 + ... + \alpha r^{k-1} = \frac{\alpha (r^k - 1)}{r-1}$...... (1)

Now let us prove that p (k + 1) is true

Here

$\{\alpha + \alpha r + \alpha r^2 + \cdots .. + \alpha rk - 1\} \ \alpha^{(k+1)-1}$

By using equation (1)

$\frac{\alpha (r^k - 1)}{r-1} + \alpha r^k$

Taking L.C.M.

$= \frac{\alpha (r^k - 1) + \alpha r^k (r-1)}{r-1}$

Multiplying the terms

$= \frac{\alpha (r^k - 1) + \alpha r^{k+1} - ar^k}{r-1}$

So we get

$= \frac{ar^k - \alpha + \alpha r^{k+1} - \alpha r^k}{r-1}$

By further simplification

$= \frac{\alpha r^{k+1} - \alpha}{r-1}$

Taking the common terms out

$= \frac{\alpha (r^{k+1} - \alpha)}{r-1}$

P (k + 1) is true whenever P (k) is true

Therefore by the principal of mathematical induction, statement P(n) is true for all natural number i.e. n

Question 13

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\ldots\left(1+\frac{(2n+1)}{n^2}\right) = (n+1)^2$$

Answer:

We can write the given statement as

P (n): $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\ldots\left(1+\frac{(2n+1)}{n^2}\right) = (n+1)^2$

If n = 1 we get

P (1): $= \left(1+\frac{3}{1}\right) = 4 = (1+1)^2 = 2^2 = 4$

Which is true?

Consider P (k) be true for some positive integer k

$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\ldots\left(1+\frac{(2k+1)}{k^2}\right) = (k+1)^2$ …… (1)

Now let us prove that p (k + 1) is true

Here

$\left[\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\ldots\left(1+\frac{(2k+1)}{k^2}\right)\right]\left\{1+\frac{\{2(k+1)+1\}}{(k+1)^2}\right\}$

By using equation (1)

$= (k+1)^2\left(1+\frac{2(k+1)+1}{(k+1)^2}\right)$

Taking LCM

$= (k+1)\left[\frac{(k+1)^2+2(k+1)+1}{(k+1)^2}\right]$

So we get

$= (k+1)^2 + 2(k+1) + 1$

By further simplification

$=\{(k+1)+1\}^2$

P (k + 1) is true whenever P (k) is true

Therefore by the principal of mathematical induction, statement P(n) is true for all natural number i.e. n

Question 14

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\ldots\left(1+\frac{1}{n}\right) = (n+1)$$

Answer:

We can write the given statement as

P (n): $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\ldots\left(1+\frac{1}{n}\right)=(n+1)$

If n = 1 we get

P (1): = $\left(1+\frac{1}{1}\right)= 2 = (1+1)$

Which is true?

Consider P (k) be true for some positive integer k

p (k): $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\ldots\left(1+\frac{1}{k}\right)=(k+1)$ ……. (1)

Now let us prove that p (k + 1) is true

Here

$\left[\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\ldots\left(1+\frac{1}{k}\right)\right]\left(1+\frac{1}{k+1}\right)$

By using equation (1)

$= (k + 1)\left(1+\frac{1}{k+1}\right)$

Taking LCM

$= (k + 1)\left(\frac{(K+1)+1}{(K+1)}\right)$

By further simplification

= (k + 1) + 1

P (k + 1) is true whenever P (k) is true

Therefore by the principal of mathematical induction, statement P(n) is true for all natural number i.e. n

Question 15

$$\mathbf{1^2+3^2+5^2+\cdots+(2n-1)^2=\frac{n\,(2n-1)(2n+1)}{3}}$$

Answer:

We can write the given statement as

P (n): $1^2+3^2+5^2+\cdots+(2n-1)^2=\frac{n\,(2n-1)(2n+1)}{3}$

If n = 1 we get

P (1): $= 1^2 = 1\ \frac{1\,(2.1-1)(2.1+1)}{3}=\frac{1.1.3}{3}=1$

Which is true?

Consider P (k) be true for some positive integer k

p(k): $1^2+3^2+5^2+\cdots+(2k-1)^2=\frac{k\,(2k-1)(2k+1)}{3}$ ……… (1)

Now let us prove that p (k + 1) is true

Here

$\{1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2\} + \{2(k+1)\}^2$

By using equation (1)

$= \frac{k(2k-1)(2k+1)}{3}(2k+2-1)^2$

So we get

$= \frac{k(2k-1)(2k+1)}{3}(2k+1)^2$

Taking LCM

$= \frac{k(2k+1)(2k+1)+3(2k+1)^2}{3}$

Taking the common terms out

$= \frac{(2k+1)\{k(2k-1)+3(2k+1)\}}{3}$

By further simplification

$= \frac{(2k+1)\{2k^2-k+6k+3\}}{3}$

So we get

$= \frac{(2k+1)\{2k^2+2k+5k+3\}}{3}$

We can write it as

$= \frac{(2k+1)\{2k^2+2k+3k+3\}}{3}$

Splitting the terms

$= \frac{(2k+1)\{2k(k+1)+(k+1)\}}{3}$

We get

$= \frac{(2k+1)(k+1)(2k+3)}{3}$

$= \frac{(k+1)\{2(k+1)-1\}\{2(k+1)+1\}}{3}$

P (k + 1) is true whenever P (k) is true

Therefore by the principal of mathematical induction, statement P(n) is true for all natural number i.e. n

Question 16

$$\frac{1}{1.4}+\frac{1}{1.4}+\frac{1}{7.4}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}$$

Answer:

We can write the given statement as

P (n): $\frac{1}{1.4}+\frac{1}{1.4}+\frac{1}{7.4}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}$

If n = 1 we get

P (1): $= \frac{1}{1.4}\,\frac{1}{3.1+1} = \frac{1}{4} = \frac{1}{1.4}$

Which is true?

Consider P (k) be true for some positive integer k

p(k): $\frac{1}{1.4}+\frac{1}{1.4}+\frac{1}{7.10}+....+\frac{1}{(3k-2)(3k+1)}=\frac{k}{(3k+1)}$ .....(1)

Now let us prove that p (k + 1) is true

Here

$\left\{\frac{1}{1.4}+\frac{1}{1.4}+\frac{1}{7.10}+\cdots.+\frac{1}{(3k-2)(3k+1)}\right\}+\frac{1}{\{3(k+1)-2\}\{3(k+1)+1\}}$

By using equation (1)

$=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}$

So we get

$=\frac{1}{(3k+1)}\left\{k+\frac{1}{(3k+4)}\right\}$

Taking LCM

$=\frac{1}{(3k+1)}\left\{\frac{k(3k+4)+1}{(3k+4)}\right\}$

Multiplying the terms

$=\frac{1}{(3k+1)}\left\{\frac{3k^2+4k+1}{(3k+4)}\right\}$

It can be written as

$=\frac{1}{(3k+1)}\left\{\frac{3k^2+3k+1}{(3k+4)}\right\}$

Separating the terms

$=\frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$

By further calculation

$=\frac{(k+1)}{3(k+1)+1}$

P (k + 1) is true whenever P (k) is true

Therefore by the principal of mathematical induction, statement P(n) is true for all natural number i.e. n

Question 17

$$\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+...+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$$

Answer:

We can write the given statement as

P (n): $\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+...+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$

If n = 1 we get

P (1): $=\frac{1}{1.3}=\frac{1}{3(2.1+3)}=\frac{1}{3.5}$

Which is true?

Consider P (k) be true for some positive integer k

p(k): $\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots+\frac{1}{(2k+1)(2k+3)}=\frac{k}{3\,(2k+3)}$ ………… (i)

Now let us prove that p (k + 1) is true

Here

$\left[\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ \ldots+\frac{1}{(2k+1)(2k+3)}\right]+\frac{1}{\{2\,(k+1)+1\}\{2\,(k+1)+3\}}$

By using equation (1)

$=\frac{k}{3\,(2\,k+3)}+\frac{1}{(2\,k+3)(2\,6+5)}$

So we get

$=\frac{k}{(2\,k+3)}\left[\frac{k}{3}+\frac{1}{(2\,k+5)}\right]$

Taking LCM

$=\frac{1}{(2\,k+3)}\left[\frac{K\,(2K+5)+3}{3\,(2K+5)}\right]$

Multiplying the terms

$=\frac{1}{(2\,k+3)}\left[\frac{2k^2+5k+3}{3\,(2k+5)}\right]$

It can be written as

$=\frac{1}{(2\,k+3)}\left[\frac{2k^2+2k+3k+3}{3\,(2k+5)}\right]$

Separating the terms

$=\frac{1}{(2\,k+3)}\left[\frac{2k\,(k+1)+3\,(k+1)}{3\,(2k+5)}\right]$

By further calculation

$=\frac{(k+1)\,(2k+3)}{3(2k+3)\,(2k+5)}$

$=\frac{(k+1)}{3\,\{2(k+1)+3\}}$

P (k + 1) is true whenever P (k) is true

Therefore by the principal of mathematical induction, statement P(n) is true for all natural number i.e. n

Question 18

1+2+3+…..+ n <$\frac{1}{8}(2n+1)^2$

Answer:

We can write the given statement as

P (n): 1+2+3+…..+ n <$\frac{1}{8}(2n+1)^2$

If n = 1 we get

P (1): = 1 <$\frac{1}{8}(2n+1)^2=\frac{9}{8}$

Which is true?

Consider P (k) be true for some positive integer k
$1+2+.....+ k < \frac{1}{8}(2k+1)^2$ …….. (1)
Now let us prove that p (k + 1) is true
Here
$(1+ 2 +......+ k) + (k + 1) < \frac{1}{8}(2k+1)^2 + (k+1)$
By using equation (1)
$< \frac{1}{8}\{(2k+1)^2 + 8(k+1)\}$
Expanding terms using formula
$< \frac{1}{8}\{4k^2 + 4k + 1 + 8(k+8)\}$
By further calculation.
$< \frac{1}{8}\{4k^2 + 12k + 9\}$
So we get
$< \frac{1}{8}(2k+3)^2$
$< \frac{1}{8}\{2(k+1) + 1\}^2$
$(1 + 2 + 3+.... +k) + (k+1) < \frac{1}{8}(2k+1)^2 + (k+1)$
P (k + 1) is true whenever P (k) is true
Therefore by the principal of mathematical induction, statement P(n) is true for all natural number i.e. n

Question 19

n (n + 1) (n + 5) is a multiple of 3

Answer:

We can write the given statement as
P (n): n (n + 1) (n + 5), which is a multiple of 3
If n = 1 we get
1 (1 + 1) (1 + 5) = 12, which is a multiple of 3
Which is true?
Consider P (k) be true for some positive integer k
k (k + 1) (k + 5) is a multiple of 3
k (k + 1) (k + 5) = 3m, where $m \in N$ …… (1)
Now let us prove that P (k + 1) is true.
Here
(k + 1) {(k + 1) + 1} {(k + 1) + 5}
We can write it as
= (k + 1) (k + 2) {(k + 5) + 1}

By multiplying the terms
= (k + 1) (k + 2) (k + 5) + (k + 1) (k + 2)
So we get
= {k (k + 1) (k + 5) + 2 (k + 1) (k + 5)} + (k + 1) (k + 2)
Substituting equation (1)
= 3m + (k + 1) {2 (k + 5) + (k + 2)}
By multiplication
= 3m + (k + 1) {2k + 10 + k + 2}
On further calculation
= 3m + (k + 1) (3k + 12)
Taking 3 as common
= 3m + 3 (k + 1) (k + 4)
We get
= 3 {m + (k + 1) (k + 4)}
= 3 × q where q = {m + (k + 1) (k + 4)} is some natural number
(k + 1) {(k + 1) + 1} {(k + 1) + 5} is a multiple of 3
P (k + 1) is true whenever P (k) is true.
Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers i.e. n.

Question 20

$102^{n-1} + 1$ is divisible by 11

Answer:

We can write the given statement as
P (n): $102^{n} - 1 + 1$ is divisible by 11
If n = 1 we get
P (1) =$102^{2.1} - 1 + 1$ = 11, which is divisible by 11
Which is true?
Consider P (k) be true for some positive integer k
$10^{2k} - 1 + 1$ is divisible by 11
$10^{2k} - 1 + 1 = 11m$, where m ∈ N …… (1)
Now let us prove that P (k + 1) is true.
Here
$10^{2(k+1)-1} + 1$
We can write it as
= $10^{2(k+1)-1} + 1$
= $10^{2(k+1)-1} + 1$
By addition and subtraction of 1
= $10^{2} (10^{2k-1} + 1 - 1) + 1$
We get
= $10^{2} (10^{2k-1} + 1) - 10^{2} + 1$

Using equation 1 we get
$= 10^2 . 11m – 100 + 1$
$= 100 \times 11m – 99$
Taking out the common terms
$= 11 (100m – 9)$
= 11 r, where r = (100m – 9) is some natural number
$10^{2(k+1)-1} + 1$ is divisible by 11
P (k + 1) is true whenever P (k) is true.
Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers i.e. n.

Question 21

$x^{2n} – y^{2n}$ is divisible by x + y

Answer:

We can write the given statement as
P (n): $x^{2n} – y^{2n}$ is divisible by x + y
If n = 1 we get
P (1) = $x^{2\times1} - y^{2\times1} = x^2 - y^2 = (x + y)(x - y)$, which is divisible by (x + y)
Which is true?
Consider P (k) be true for some positive integer k
$x^{2n} – y^{2n}$ is divisible by x + y
$x^{2k} – y^{2k} = m (x + y)$, where m ∈ N …… (1)
Now let us prove that P (k + 1) is true.
Here
$x^{2(k+1)} – y^{2(k+1)}$
We can write it as
$= x^{2k} . x^2 – y^{2k} . y^2$
By adding and subtracting y2k we get
$= x^2 (x^{2k} – y^{2k} + y^{2k}) – y^{2k} . y^2$
From equation (1) we get
$= x^2 \{m (x + y) + y^{2k}\} - y^{2k} . y^2$
By multiplying the terms
$= m (x + y) x^2 + y^{2k} . x^2$ – y2k. y2
Taking out the common terms
$= m (x + y) x^2 + y^{2k} (x^2 – y^2)$
Expanding using formula
$= m (x + y) x^2 + y^{2k} (x + y) (x – y)$
So we get
$= (x + y) \{mx^2 + y^{2k} (x – y)\}$, which is a factor of (x + y)
P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers i.e. n.

Question 22

$3^{2n+2} – 8n – 9$ is divisible by 8

Answer:

We can write the given statement as
P (n): $3^{2n+2} – 8n – 9$ is divisible by 8
If n = 1 we get
P (1) = $3^{2\times1+2} – 8 \times 1 – 9 = 64$, which is divisible by 8
Which is true?
Consider P (k) be true for some positive integer k
$3^{2\times1+2} – 8k – 9$ is divisible by 8
32k + 2 – 8k – 9 = 8m, where m ∈ N …… (1)
Now let us prove that P (k + 1) is true.
Here
3 2(k + 1) + 2 – 8 (k + 1) - 9
We can write it as
= $3 . 3^{2k+2} – 8k – 8 - 9$
By adding and subtracting 8k and 9 we get
= $3^2 (3^{2k+2} – 8k – 9 + 8k + 9) – 8k – 17$
On further simplification
= $32 (3^{2\times1+2} – 8k – 9) + 32 (8k + 9) – 8k - 17$
From equation (1) we get
= 9. 8m + 9 (8k + 9) – 8k - 17
By multiplying the terms
= 9. 8m + 72k + 81 – 8k - 17
So we get
= 9. 8m + 64k + 64
By taking out the common terms
= 8 (9m + 8k + 8)
= 8r, where r = (9m + 8k + 8) is a natural number
So $3^{2(k+1)+2} – 8 (k + 1) – 9$ is divisible by 8
P (k + 1) is true whenever P (k) is true.
Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers i.e. n.

Question 23

$41^n – 14^n$ is a multiple of 27

Answer:

We can write the given statement as
P (n):$41^n – 14^n$ is a multiple of 27
If n = 1 we get
P (1) = 411 - 141 = 27, which is a multiple by 27
Which is true?
Consider P (k) be true for some positive integer k
$41^k – 14k$is a multiple of 27
$41^k – 14^k = 27m$, where m ∈ N …… (1)
Now let us prove that P (k + 1) is true.
Here
$41^{k + 1} – 14^{k + 1}$
We can write it as
$= 41^k. 41 – 14^k. 14$
By adding and subtracting 14k we get
$= 41 (41^k – 14^k + 14^k) – 14^k. 14$
On further simplification
$= 41 (41^k– 14^k) + 41. 14^k – 14^k. 14$
From equation (1) we get
$= 41. 27m + 14^k (41 – 14)$
By multiplying the terms
$= 41. 27m + 27. 14^k$
By taking out the common terms
$= 27 (41m – 14^k)$
= 27r, where r = (41m – 14k) is a natural number
So $41^{k+1}– {}^{14k+1}$ is a multiple of 27
P (k + 1) is true whenever P (k) is true.
Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers i.e. n.

Question 24

$(2n +7) < (n + 3)^2$

Answer:

We can write the given statement as
P(n): $(2n +7) < (n + 3)^2$
If n = 1 we get
$2.1 + 7 = 9 < (1 + 3)^2 = 16$
Which is true?
Consider P (k) be true for some positive integer k
$(2k + 7) < (k + 3)^2$ … (1)

Now let us prove that P (k + 1) is true.
Here
{2 (k + 1) + 7} = (2k + 7) + 2
We can write it as
= {2 (k + 1) + 7}
From equation (1) we get
(2k + 7) + 2 < (k + 3)2 + 2
By expanding the terms
2 (k + 1) + 7 < k^2 + 6k + 9 + 2
On further calculation
2 (k + 1) + 7 < k^2 + 6k + 11
Here k^2 + 6k + 11 < k^2 + 8k + 16
We can write it as
2 (k + 1) + 7 < $(k + 4)^2$
2 (k + 1) + 7 < $\{(k + 1) + 3\}^2$
P (k + 1) is true whenever P (k) is true.
Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers i.e. n.