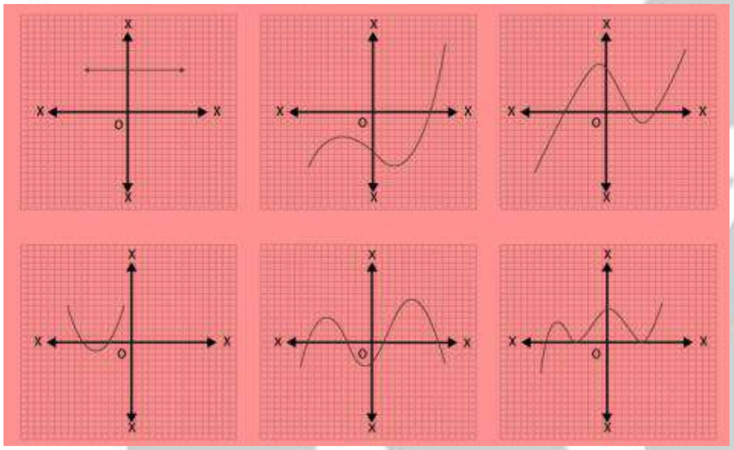
Chapter 2
Polynomials

Exercise 2.1

Question 1

The graphs of y = p(x) is given in Fig. 2.10 below, for some polynomials p(x). Find the number of zeroes of p(x), in each case.



Solutions:

Graphical method to find zeroes:

Total number of zeroes in any polynomial equation = total number of times the curve intersects x-axis.

I. In the given graph, the number of zeroes of p(x) is 0 because the graph is parallel to x-axis does not cut it at any point.
II. In the given graph, the number of zeroes of p(x) is 1 because the graph intersects the x-axis at only one point.
III. In the given graph, the number of zeroes of p(x) is 3 because the graph intersects the x-axis at any three points.
IV. In the given graph, the number of zeroes of p(x) is 2 because the graph intersects the x-axis at two points.
V. In the given graph, the number of zeroes of p(x) is 4 because the graph intersects the x-axis at four points.
VI. In the given graph, the number of zeroes of p(x) is 3 because the graph intersects the x-axis at three points.

Exercise 2.2

Question 1

Find the zeroes of the following quadratic polynomials and verify the relationship between the zeroes and the coefficients.

Solutions:

(i)$x^2–2x–8$
$\Rightarrow x^2– 4x+2x–8 = x(x–4) +2(x–4) = (x-4)(x+2)$
Therefore, zeroes of polynomial equation $x^2–2x–8$ are (4, -2)
Sum of zeroes = 4–2 = 2 = -(-2)/1 = -(Coefficient of x)/ (Coefficient of x^2)
Product of zeroes = 4× (-2) = -8 =-(8)/1 = (Constant term)/ (Coefficient of x^2)

(ii)$4s^2–4s+1$
$\Rightarrow 4s^2–2s–2s+1 = 2s(2s–1)–1(2s-1) = (2s–1)(2s–1)$
Therefore, zeroes of polynomial equation $4s^2–4s+1$ are (1/2, 1/2)
Sum of zeroes = (½)+(1/2) = 1 = -4/4 = -(Coefficient of s)/ (Coefficient of s^2)
Product of zeros = (1/2) × (1/2) = 1/4 = (Constant term)/ (Coefficient of s^2)

(iii) $6x^2–3–7x$
$\Rightarrow 6x^2–7x–3 = 6x^2 – 9x + 2x – 3 = 3x(2x - 3) +1(2x - 3) = (3x+1)(2x-3)$
Therefore, zeroes of polynomial equation $6x^2–3–7x$ are (-1/3, 3/2)
Sum of zeroes = -(1/3) +(3/2) = (7/6) = -(Coefficient of x)/ (Coefficient of x^2)
Product of zeroes = -(1/3) × (3/2) = -(3/6) = (Constant term) / (Coefficient of x^2)

(iv)$4u^2+8u$
$\Rightarrow 4u(u+2)$
Therefore, zeroes of polynomial equation $4u^2 + 8u$ are (0, -2).
Sum of zeroes = 0+(-2) = -2 = -(8/4) = = -(Coefficient of u)/ (Coefficient of u^2)
Product of zeroes = 0×-2 = 0 = 0/4 = (Constant term)/ (Coefficient of u^2)

(v) $t^2–15$
$\Rightarrow t^2 = 15$ or $t = \pm\sqrt{15}$
Therefore, zeroes of polynomial equation $t^2 –15$ are ($\sqrt{15}$, -$\sqrt{15}$)
Sum of zeroes =$\sqrt{15}$+(-$\sqrt{15}$) = 0= -(0/1) = -(Coefficient of t) / (Coefficient of t^2)
Product of zeroes = $\sqrt{15}$× (-$\sqrt{15}$) = -15 = -15/1 = (Constant term) / (Coefficient of t^2)

(vi) $3x^2–x–4$
$\Rightarrow 3x^2–4x+3x–4 = x(3x-4) +1(3x-4) = (3x - 4)(x + 1)$
Therefore, zeroes of polynomial equation$3x^2 – x – 4$ are (4/3, -1)
Sum of zeroes = (4/3) +(-1) = (1/3) = -(-1/3) = -(Coefficient of x) / (Coefficient of x^2)

Product of zeroes= (4/3) × (-1) = (-4/3) = (Constant term) / (Coefficient of x^2)

Question 2

Find a quadratic polynomial each with the given numbers as the sum and product of its zeroes respectively.

(i) 1/4, -1

Solution:

From the formulas of sum and product of zeroes, we know,
Sum of zeroes = $\alpha+\beta$
Product of zeroes = $\alpha \beta$
Sum of zeroes = $\alpha+\beta$ = 1/4
Product of zeroes = $\alpha \beta$ = -1
∴ If α and β are zeroes of any quadratic polynomial, then the quadratic polynomial equation can bewritten directly as:
$x^2-(\alpha+\beta) x +\alpha\beta = 0$
$x^2- (1/4) x +(-1) = 0$
$4x^2-x-4 = 0$
Thus,$4x^2-x-4$ is the quadratic polynomial.

(ii)$\sqrt{2}$, 1/3

Solution:

Sum of zeroes = $\alpha + \beta =\sqrt{2}$
Product of zeroes = $\alpha \beta$ = 1/3
∴ If α and β are zeroes of any quadratic polynomial, then the quadratic polynomial equation can be written directly as:
$x^2-(\alpha+\beta) x +\alpha\beta = 0$
$x^2- (\sqrt{2}) x + (1/3) = 0$
$3x^2 -3\sqrt{2}x+1 = 0$
Thus, $3x^2 -3\sqrt{2}x+1$ is the quadratic polynomial.

(iii) 0, $\sqrt{5}$

Solution:

Given,
Sum of zeroes = $\alpha+\beta$ = 0
Product of zeroes = $\alpha \beta = \sqrt{5}$
∴ If α and β are zeroes of any quadratic polynomial, then the quadratic polynomial equation can be written directly as:

$x^2-(\alpha+\beta)\ x + \alpha\beta = 0$
$x^2- (0)x + \sqrt{5} = 0$
Thus, x 2+√5 is the quadratic polynomial.

(iv) 1, 1

Solution:

Given,
Sum of zeroes = $\alpha+\beta = 1$
Product of zeroes = $\alpha\ \beta = 1$
∴ If α and β are zeroes of any quadratic polynomial, then the quadratic polynomial equation can be written directly as:
$x^2-(\alpha+\beta)\ x + \alpha\beta = 0$
$x^2-x+1 = 0$
Thus, x2–x+1is the quadratic polynomial.

(v) -1/4, 1/4

Solution:

Given,
Sum of zeroes = $\alpha+\beta = -1/4$
Product of zeroes = $\alpha\ \beta = 1/4$
∴ If α and β are zeroes of any quadratic polynomial, then the quadratic polynomial equation can be written directly as:
$x^2-(\alpha+\beta)\ x + \alpha\beta = 0$
$x^2- (-1/4)\ x + (1/4) = 0$
$4x^2+x+1 = 0$
Thus,$4x^2+x+1$ is the quadratic polynomial.

(vi) 4, 1

Solution:

Given,
Sum of zeroes = $\alpha+\beta = 4$
Product of zeroes = $\alpha\beta = 1$
∴ If α and β are zeroes of any quadratic polynomial, then the quadratic polynomial equation can bewritten directly as:
$x^2-(\alpha+\beta)\ x+\alpha\beta = 0$
$x^2-4x+1 = 0$
Thus,x^2-4x+1 is the quadratic polynomial.

Exercise 2.3

Question 1

Divide the polynomial p(x) by the polynomial g(x) and find the quotient and remainder in each of the following:
(i) $p(x) = x^3 - 3x^2 + 5x - 3$, $g(x) = x^2 - 2$

Solution:

Given,
Dividend = $p(x) = x^3 - 3x^2 + 5x - 3$
Divisor = $g(x) = x^2 - 2$

$$
\begin{array}{r|rrrr}
 & x & -3 & & \\
x^2-2 & x^3 & -3x^2 & +5x & -3 \\
 & - & & & \\
 & x^3 & +0x^2 & -2x & \\
\hline
 & & -3x^2 & +7x & -3 \\
 & & - & & \\
 & & -3x^2 & +0x & +6 \\
\hline
 & & & 7x & -9
\end{array}
$$

Therefore, upon division we get,
Quotient = $x - 3$
Remainder = $7x - 9$

(ii) $p(x) = x^4 - 3x^2 + 4x + 5$, $g(x) = x^2 + 1 - x$

Solution:

Given, Dividend = $p(x) = x^4 - 3x^2 + 4x + 5$
Divisor = $g(x) = x^2 + 1 - x$

$$
\begin{array}{r|rrrrr}
 & x^2 & +x & -3 & & \\
\hline
x^2 - x + 1 & x^4 & +0x^3 & -3x^2 & +4x & +5 \\
- & x^4 & -x^3 & +x^2 & & \\
\hline
 & & x^3 & -4x^2 & +4x & +5 \\
 & - & x^3 & -x^2 & +x & \\
\hline
 & & & -3x^2 & +3x & +5 \\
 & & - & -3x^2 & +3x & -3 \\
\hline
 & & & & & 8
\end{array}
$$

Therefore, upon division we get,
Quotient = $x^2 + x - 3$
Remainder = 8

(iii) p(x) =$x^4 - 5x + 6$, g(x) = $2 - x^2$

Solution:

Given,
Dividend = p(x) =$x^4 - 5x + 6 = x^4 + 0x^2 - 5x + 6$
Divisor = g(x) = $2 - x^2 = -x^2 + 2$

$$
\begin{array}{r|rrrrr}
 & -x^2 & -2 & & & \\
\hline
-x^2 + 2 & x^4 & +0x^3 & +0x^2 & -5x & +6 \\
- & x^4 & +0x^3 & -2x^2 & & \\
\hline
 & & & 2x^2 & -5x & +6 \\
 & & - & 2x^2 & +0x & -4 \\
\hline
 & & & & -5x & +10
\end{array}
$$

Therefore, upon division we get,
Quotient = $-x^2 - 2$
Remainder = $-5x + 10$

Question 2

Check whether the first polynomial is a factor of the second polynomial by dividing the second polynomial by the first polynomial:
(i) t^2 -3, $2t^4$ +$3t^3$ -$2t^2$ -9t-12

Solutions:

Given, First polynomial = $t^2 -3$
Second polynomial = $2t^4 +3t^3 -2t^2 -9t-12$

$$
\begin{array}{rrrrrr}
 & 2t^2 & +3t & +4 & & \\
\cline{2-6}
t^2-3 & \big)\,2t^4 & +3t^3 & -2t^2 & -9t & -12 \\
 & - & & & & \\
 & 2t^4 & +0t^3 & -6t^2 & & \\
\cline{2-6}
 & & 3t^3 & +4t^2 & -9t & -12 \\
 & & - & & & \\
 & & 3t^3 & +0t^2 & -9t & \\
\cline{2-6}
 & & & 4t^2 & +0t & -12 \\
 & & & - & & \\
 & & & 4t^2 & +0t & -12 \\
\cline{2-6}
 & & & & & 0
\end{array}
$$

As we can see, the remainder is left as 0. Therefore, we say that, t^2 -3 is a factor of $2t^2+3t+4$.

(ii)x^2+3x+1, $3x^4+5x^3-7x^2+2x+2$

Solutions:

Given,
First polynomial = x^2+3x+1
Second polynomial = $3x^4+5x^3-7x^2+2x+2$

$$
\begin{array}{r|l}
 & 3x^2 \quad -4x \quad +2 \\
\hline
x^2+3x+1 & 3x^4 \quad +5x^3 \quad -7x^2 \quad +2x \quad +2 \\
 & - \\
 & 3x^4 \quad +9x^3 \quad +3x^2 \\
\hline
 & \qquad -4x^3 \quad -10x^2 \quad +2x \quad +2 \\
 & \qquad - \\
 & \qquad -4x^3 \quad -12x^2 \quad -4x \\
\hline
 & \qquad\qquad 2x^2 \quad +6x \quad +2 \\
 & \qquad\qquad - \\
 & \qquad\qquad 2x^2 \quad +6x \quad +2 \\
\hline
 & \qquad\qquad\qquad 0
\end{array}
$$

As we can see, the remainder is left as 0. Therefore, we say that, $x^2 + 3x + 1$ is a factor of $3x^4+5x^37x^2+2x+2$.

(iii) x^3 -3x+1, x^5 -4x^3+x^2+3x+1

Solutions:

Given, First polynomial = x^3 -3x+1
Second polynomial = x ^{5} -4x^3+x^2+3x+1

$$
\begin{array}{r|l}
 & x^2 \quad -1 \\
\hline
x^3-3x+1 & x^5 \quad +0x^4 \quad -4x^3 \quad +x^2 \quad +3x \quad +1 \\
 & - \\
 & x^5 \quad +0x^4 \quad -3x^3 \quad +x^2 \\
\hline
 & \qquad -x^3 \quad +0x^2 \quad +3x \quad +1 \\
 & \qquad - \\
 & \qquad -x^3 \quad +0x^2 \quad +3x \quad -1 \\
\hline
 & \qquad\qquad\qquad 2
\end{array}
$$

As we can see, the remainder is not equal to 0. Therefore, we say that, x^3 -3x+1 is not a factor of x ^{5} -4x^3+x^2+3x+1.

Question 3

Obtain all other zeroes of $3x^4+6x^3-2x^2-10x-5$, if two of its zeroes are $\sqrt{(5/3)}$ and $-\sqrt{(5/3)}$.

Solutions:

Since this is a polynomial equation of degree 4, hence there will be total 4 roots.
$\sqrt{(5/3)}$ and $-\sqrt{(5/3)}$ are zeroes of polynomial f(x).
$\therefore (x-\sqrt{(5/3)})(x+\sqrt{(5/3)} = x\ 2 -(5/3) = 0$
$(3x^2-5)=0$, is a factor of given polynomial f(x).
Now, when we will divide f(x) by $(3x^2-5)$ the quotient obtained will also be a factor of f(x) and the remainder will be 0.

$$
\begin{array}{r|l}
 & x^2 + 2x + 1 \\
\hline
3x^2-5 & 3x^4 + 6x^3 - 2x^2 - 10x - 5 \\
 & 3x^4 \quad - 5x^2 \\
 & (-) \quad (+) \\
\hline
 & +6x^3 + 3x^2 - 10x - 5 \\
 & -6x^3 \quad - 10x \\
 & (+) \quad (-) \\
\hline
 & 3x^2 \quad -5 \\
 & 3x^2 \quad -5 \\
 & (-) \quad (+) \\
\hline
 & 0
\end{array}
$$

Therefore, $3x^4+6x^3-2x^2-10x-5 = (3x^2-5)(x^2+2x+1)$
Now, on further factorizing (x^2+2x+1) we get,
$x^2+2x+1 = x^2+x+x+1 = 0$
$x(x+1)+1(x+1) = 0$
$(x+1)(x+1) = 0$
So, its zeroes are given by: $x=-1$ and $x=-1$.
Therefore, all four zeroes of given polynomial equation are:
$\sqrt{(5/3)}, -\sqrt{(5/3)}, -1$ and -1.
Hence, is the answer.

Question 4

On dividing x^3-3x^2+x+2 by a polynomial g(x), the quotient and remainder were $x-2$ and $-2x+4$, respectively. Find g(x).

Solutions:

Given,
Dividend, $p(x) = x^3 -3x^2+x+2$
Quotient = $x-2$
Remainder = $-2x+4$
We have to find the value of Divisor, $g(x) =?$
As we know,
Dividend = Divisor × Quotient + Remainder
$\therefore x^3 -3x^2+x+2 = g(x)\times(x-2) + (-2x+4)$
$x^3 -3x^2+x+2-(-2x+4) = g(x)\times(x-2)$
Therefore, $g(x) \times (x-2) = x^3 -3x^2+x+2$
Now, for finding $g(x)$ we will divide $x^3 -3x^2+x+2$ with $(x-2)$

$$
\begin{array}{r|l}
 & x^2 - x + 1 \\
\hline
x - 2 & x^3 - 3x^2 + 3x - 2 \\
 & x^3 - 2x^2 \\
 & (-) \quad (+) \\
\hline
 & -x^2 + 3x - 2 \\
 & -x^2 + 2x \\
 & (+) \quad (-) \\
\hline
 & x - 2 \\
 & x - 2 \\
 & (-) \quad (+) \\
\hline
 & 0 \\
\hline
\end{array}
$$

Therefore, $g(x) = (x^2-x+1)$

Question 5

Give examples of polynomials p(x), g(x), q(x) and r(x), which satisfy the division algorithm and
(i) deg p(x) = deg q(x)
(ii) deg q(x) = deg r(x)
(iii) deg r(x) = 0

Solutions:

According to the division algorithm, dividend p(x) and divisor g(x) are two polynomials, where g(x)≠0. Then wecan find the value of quotient q(x) and remainder r(x), with the help of below given formula;
Dividend = Divisor × Quotient + Remainder
∴ p(x) = g(x)×q(x)+r(x)
Where r(x) = 0 or degree of r(x)< degree of g(x).
Now let us proof the three given cases as per division algorithm by taking examples for each.

(i) deg p(x) = deg q(x)
Degree of dividend is equal to degree of quotient, only when the divisor is a constant term.
Let us take an example, p(x) = $3x^2 + 3x + 3$ is a polynomial to be divided by g(x) = 3.
So, $(3x^2 + 3x + 3)/3 = x^2 + x + 1 = q(x)$
Thus, you can see, the degree of quotient q(x) = 2, which also equal to the degree of dividend p(x).
Hence, division algorithm is satisfied here.

(ii) deg q(x) = deg r(x)
Let us take an example, p(x) = x2 + 3 is a polynomial to be divided by g(x) = x - 1.
So, $x^2 + 3 = (x - 1) \times (x) + (x + 3)$
Hence, quotient q(x) = x
Also, remainder r(x) = x + 3
Thus, you can see, the degree of quotient q(x) = 1, which is also equal to the degree of remainder r(x).
Hence, division algorithm is satisfied here.

(iii) deg r(x) = 0
The degree of remainder is 0 only when the remainder left after division algorithm is constant.
Let us take an example, p(x) = $x^2 + 1$ is a polynomial to be divided by g(x) = x.
So, $x^2 + 1 = (x)\times(x) + 1$
Hence, quotient q(x) = x
And, remainder r(x) = 1
Clearly, the degree of remainder here is 0.
Hence, division algorithm is satisfied here

Exercise 2.4

Question 1

Verify that the numbers given alongside of the cubic polynomials below are their zeroes. Also verify the relationship between the zeroes and the coefficients in each case:

(i) $2x^3+x^2 -5x+2$; -1/2, 1, -2

Solution:

Given, p(x) = $2x^3+x^2 -5x+2$
And zeroes for p(x) are = 1/2, 1, -2

∴p (1/2) = $2(1/2)^3+(1/2)2-5(1/2)+2 = (1/4)+(1/4)-(5/2)+2 = 0$
p (1) = $2(1)^3+(1)^2-5(1)+2 = 0$
p (-2) = $2(-2)^3+(-2)^2-5(-2)+2 = 0$
Hence, proved 1/2, 1, -2 are the zeroes of $2x^3+x^2-5x+2$.
Now, comparing the given polynomial with general expression, we get;
∴ $ax^3+bx^2+cx+d = 2x^3+x^2-5x+2$
a=2, b=1, c= -5 and d = 2
As we know, if α, β, γ are the zeroes of the cubic polynomial ax^3+bx^2+cx+d, then;
$\alpha+\beta+\gamma = -b/a$
$\alpha\beta+\beta\gamma+\gamma\alpha = c/a$
$\alpha\beta\gamma = -d/a$.
Therefore, putting the values of zeroes of the polynomial,
$\alpha+\beta+\gamma = ½+1+(-2) = -1/2 = -b/a$
$\alpha\beta+\beta\gamma+\gamma\alpha = (1/2\times1)+(1\times-2)+(-2\times1/2) = -5/2 = c/a$
$\alpha\beta\gamma = ½\times1\times(-2) = -2/2 = -d/a$
Hence, the relationship between the zeroes and the coefficients are satisfied.

(ii) x^3-4x^2+5x-2 ;2, 1, 1

Solution:

Given, p(x) = x^3-4x^2+5x-2
And zeroes for p(x) are 2,1,1.
∴p (2) = $23-4(2)^2+5(2)-2 = 0$
p (1) = $1^3-(4\times12)+(5\times1)-2 = 0$
Hence proved, 2, 1, 1 are the zeroes of x^3-4x^2+5x-2
Now, comparing the given polynomial with general expression, we get;
∴ $ax^3+bx^2+cx+d = x^3-4x^2+5x-2$
a = 1, b = -4, c = 5 and d = -2
As we know, if α, β, γ are the zeroes of the cubic polynomial ax^3+bx^2+cx+d, then;
$\alpha + \beta + \gamma = -b/a$
$\alpha\beta + \beta\gamma + \gamma\alpha = c/a$
$\alpha\beta\gamma = -d/a$.
Therefore, putting the values of zeroes of the polynomial,
$\alpha+\beta+\gamma = 2+1+1 = 4 = -(-4)/1 = -b/a$
$\alpha\beta+\beta\gamma+\gamma\alpha = 2\times1+1\times1+1\times2 = 5 = 5/1 = c/a$
$\alpha\beta\gamma = 2\times1\times1 = 2 = -(-2)/1 = -d/a$
Hence, the relationship between the zeroes and the coefficients are satisfied.

Question 2

Find a cubic polynomial with the sum, sum of the product of its zeroes taken two at a time, and the product of its zeroes as 2, –7, –14 respectively.

Solution:

Let us consider the cubic polynomial is ax^3+bx^2+cx+d and the values of the zeroes of the polynomials be α, β, γ.
As per the given question,
$\alpha+\beta+\gamma = -b/a = 2/1$
$\alpha\beta +\beta\gamma+\gamma\alpha = c/a = -7/1$
$\alpha \beta\gamma = -d/a = -14/1$
Thus, from above three expressions we get the values of coefficient of polynomial.
$a = 1, b = -2, c = -7, d = 14$
Hence, the cubic polynomial is $x^3 -2x^2 -7x+14$

Question 3

If the zeroes of the polynomial $x^3 -3x^2+x+1$ are $a - b, a, a + b$, find a and b.

Solution:

We are given with the polynomial here,
$p(x) = x^3 -3x^2+x+1$
And zeroes are given as $a - b, a, a + b$
Now, comparing the given polynomial with general expression, we get;
$\therefore px^3+qx^2+rx+s = x^3 -3x^2+x+1$
$p = 1, q = -3, r = 1$ and $s = 1$
Sum of zeroes = $a - b + a + a + b$
$-q/p = 3a$
Putting the values q and p.
$-(-3)/1 = 3a$
$a=1$
Thus, the zeroes are $1-b, 1, 1+b$.
Now, product of zeroes = $1(1-b) (1+b)$
$-s/p = 1-b^2$
$-1/1 = 1-b^2$
$b^2 = 1+1 = 2$
$b = \sqrt{2}$
Hence, $1-\sqrt{2}, 1, 1+\sqrt{2}$ are the zeroes of $x^3 -3x^2+x+1$.

Question 4

If two zeroes of the polynomial $x^4 -6x^3 -26x^2+138x-35$ are $2 \pm\sqrt{3}$, find other zeroes.

Solution:

Since this is a polynomial equation of degree 4, hence there will be total 4 roots
Let $f(x) = x^4 -6x^3 -26x^2+138x-35$

Since 2 +√3 and 2-√3 are zeroes of given polynomial f(x).

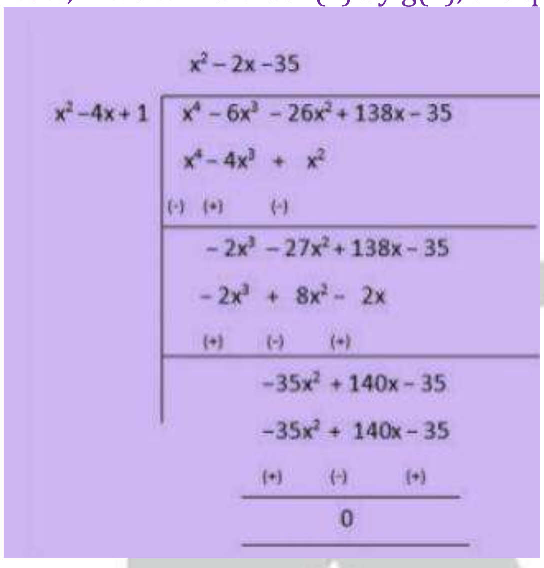
∴ [x−(2+√3)] [x−(2-√3)] = 0

(x−2−√3) (x−2+√3) = 0

On multiplying the above equation, we get,

x^2 -4x+1, this is a factor of a given polynomial f(x).

Now, if we will divide f(x) by g(x), the quotient will also be a factor of f(x) and the remainder will be 0.



So, x^4 -6x^3 -26x^2+138x-35 = (x^2 -4x+1) (x^2 −2x−35)

Now, on further factorizing (x^2−2x−35) we get,

x^2− (7−5) x −35 = x^2− 7x+5x+35 = 0

x (x −7) +5(x−7) = 0

(x+5) (x−7) = 0

So, its zeroes are given by:

x= −5 and x = 7.

Therefore, all four zeroes of given polynomial equation are: 2+√3, 2-√3, −5 and 7