Chapter 12
Introduction to Three Dimensional Geometry

Exercise 12 .1

Question 1

A point is on the x-axis. What are its y coordinate and z-coordinates?

Solution:

If a point is on the x-axis, then the coordinates of y and z are 0.
So the point is (x, 0, 0).

Question 2

A point is in the XZ-plane. What can you say about its y-coordinate?

Solution:

If a point is in XZ plane, then its y-co-ordinate is 0.

Question 3

Name the octants in which the following points lie:
(1, 2, 3), (4, –2, 3), (4, –2, –5), (4, 2, –5), (– 4, 2, –5), (– 4, 2, 5), (–3, –1, 6) (2, – 4, –7)

Solution:

Here is the table which represent the octants:

Octants	I	II	III	IV	V	VI	VII	VIII
x	+	-	-	+	+	-	-	+
y	+	+	-	-	+	+	-	-
z	+	+	+	+	-	-	-	-

(i) (1,2,3)
Here x is positive y is positive and z is positive.
So it lies in I octant.

(ii) (4, -2, 3)
Here x is positive, y is negative and z is positive.

So it lies in IV octant

(iii)(4, -2,-5)
Here x is positive, y is negative and z is negative.
So it lies in VIII octant.

(iv) (4, 2, -5)
Here x is positive, y is positive and z is negative.
So it lies in V octant.

(v) (-4, 2, -5)
Here x is negative, y is positive and z is negative.
So it lies in VI octant

(vi) (-4, 2, 5)
Here x is negative, y is positive and z is positive.
So it lies in II octant.

(Vii) (-3, -1, 6)
Here x is negative, y is negative and z is positive.
So it lies in III octant.

(viii) (2, -4, -7)
Here x is negative, y is negative and z is positive.
So it lies in III octant.

Question 4

Fill in the blanks:
(i) The x-axis and y-axis taken together determine a plane known as ______.
(ii) The coordinates of points in the XY-plane are of the form ______.
(ii) Coordinate planes divide the space into ______ octants.

Solution:

(i) The x-axis and y-axis taken together determine a plane known as XY Plane.
(ii) The coordinates of points in the XY-plane are of the form (x, y, 0).
(iii) Coordinate planes divide the space into eight octants.

Exercise 12 .2

Question 1

Find the distance between the following pairs of points:
(i) (2, 3, 5) and (4, 3, 1)
(ii) (–3, 7, 2) and (2, 4, –1)

(iii) (–1, 3, – 4) and (1, –3, 4)
(iv) (2, –1, 3) and (–2, 1, 3)

Solution:

(i) (2, 3, 5) and (4, 3, 1)
Let P be (2, 3, 5) and Q be (4, 3, 1)
By using the formula,
Distance PQ= $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$
So here,
$x_1 = 2, y_1 = 3, z_1 = 5$
$x_2 = 4, y_2 = 3, z_2 = 1$
Distance PQ = $\sqrt{[(4 - 2)^2 + (3 - 3)^2 + (1 - 5)^2]}$
$= \sqrt{[(2)^2 + (0)^2 + (-4)^2]}$
$= \sqrt{[4 + 0 + 16]}$
$= \sqrt{20}$
$= 2\sqrt{5}$
∴ The required distance is $2\sqrt{5}$ units.

(ii) (–3, 7, 2) and (2, 4, –1)
Let P be (– 3, 7, 2) and Q be (2, 4, – 1)
By using the formula,
Distance PQ = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$
So here,
$x_1 = -3, y_1 = 7, z_1 = 2$
$x_2 = 2, y_2 = 4, z_2 = -1$
Distance PQ = $\sqrt{[(2 - (-3)^2 + (4 - 7)^2 + (-1 - 2)^2]}$
$= \sqrt{[(5)^2 + (-3)^2 + (-3)^2]}$
$= \sqrt{[25 + 9 + 9]}$
$= \sqrt{43}$
∴ The required distance is $\sqrt{43}$ units.

(iii) (–1, 3, – 4) and (1, –3, 4)
Let P be (– 1, 3, – 4) and Q be (1, – 3, 4)
By using the formula,
Distance PQ = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$
So here,
$x_1 = -1, y_1 = 3, z_1 = -4$
$x_2 = 1, y_2 = 3, z_2 = 4$
Distance PQ = $\sqrt{[(1 - (-1)^2 + (-3 - 3)^2 + (4 - (-4))^2]}$
$= \sqrt{[(2)^2 + (-6)^2 + (8)^2]}$
$= \sqrt{[4 + 36 + 64]}$
$= \sqrt{104}$
$= 2\sqrt{26}$

∴ The required distance is $2\sqrt{26}$ units.

(iv) (2, –1, 3) and (–2, 1, 3)

Let P be (2, – 1, 3) and Q be (– 2, 1, 3)

By using the formula,

Distance PQ = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

So here,

$x_1 = 2, y_1 = -1, z_1 = 3$

$x_2 = -2, y_2 = 1, z_2 = 3$

Distance PQ = $\sqrt{[(-2-2)^2 + (1-(-1))^2 + (3-3)^2]}$

$= \sqrt{[(-4)^2 + (2)^2 + (0)^2]}$

$= \sqrt{[16 + 4 + 0]}$

$= \sqrt{20}$

$= 2\sqrt{5}$

∴ The required distance is $2\sqrt{5}$ units.

Question 2

Show that the points (–2, 3, 5), (1, 2, 3) and (7, 0, –1) are collinear.

Solution:

If three points are collinear, then they lie on a line.

Firstly let us calculate distance between the 3 points

i.e. PQ, QR and PR

Calculating PQ

P ≡ (– 2, 3, 5) and Q ≡ (1, 2, 3)

By using the formula,

Distance PQ = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

So here,

$x_1 = -2, y_1 = 3, z_1 = 5$

$x_2 = 1, y_2 = 2, z_2 = 3$

Distance PQ = $\sqrt{[(1-(-2))^2 + (2-3)^2 + (3-5)^2]}$

$= \sqrt{[(3)^2 + (-1)^2 + (-2)^2]}$

$= \sqrt{[9 + 1 + 4]}$

$= \sqrt{14}$

Calculating QR

Q ≡ (1, 2, 3) and R ≡ (7, 0, – 1)

By using the formula,

Distance QR = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

So here,

$x_1 = 1, y_1 = 2, z_1 = 3$

$x_2 = 7, y_2 = 0, z_2 = -1$

Distance QR = $\sqrt{[(7-1)^2 + (0-2)^2 + (-1-3)^2]}$

$= \sqrt{[(6)^2 + (-2)^2 + (-4)^2]}$

$= \sqrt{[36 + 4 + 16]}$
$= \sqrt{56}$
$= 2\sqrt{14}$

Calculating PR
P ≡ (– 2, 3, 5) and R ≡ (7, 0, – 1)
By using the formula,
Distance PR = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$
So here,
$x_1 = -2, y_1 = 3, z_1 = 5$
$x_2 = 7, y_2 = 0, z_2 = -1$
Distance PR = $\sqrt{[(7 - (-2))^2 + (0 - 3)^2 + (-1 - 5)^2]}$
$= \sqrt{[(9)^2 + (-3)^2 + (-6)^2]}$
$= \sqrt{[81 + 9 + 36]}$
$= \sqrt{126}$
$= 3\sqrt{14}$

Thus, PQ= $\sqrt{14}$, QR= $2\sqrt{14}$, and PR = $3\sqrt{14}$
So, PQ + QR = $\sqrt{14} + 2\sqrt{14}$
$= 3\sqrt{14}$
= PR
∴ The points P, Q and R are collinear.

Question 3

Verify the following:
(i) (0, 7, –10), (1, 6, – 6) and (4, 9, – 6) are the vertices of an isosceles triangle.
(ii) (0, 7, 10), (–1, 6, 6) and (– 4, 9, 6) are the vertices of a right angled triangle.
(iii) (–1, 2, 1), (1, –2, 5), (4, –7, 8) and (2, –3, 4) are the vertices of a parallelogram.

Solution:

(i) (0, 7, –10), (1, 6, – 6) and (4, 9, – 6) are the vertices of an isosceles triangle.

Let us consider the points be
P (0, 7, –10), Q (1, 6, – 6) and R (4, 9, – 6)
If any 2 sides are equal, hence it will be an isosceles triangle
So firstly let us calculate the distance of PQ, QR
Calculating PQ
P ≡ (0, 7, – 10) and Q ≡ (1, 6, – 6)
By using the formula,
Distance PQ = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$
So here,
$x_1 = 0, y_1 = 7, z_1 = -10$
$x_2 = 1, y_2 = 6, z_2 = -6$
Distance PQ= $\sqrt{[(1 - 0)^2 + (6 - 7)^2 + (-6 - (-10))^2]}$
$= \sqrt{[(1)^2 + (-1)^2 + (4)^2]}$

$= \sqrt{[1 + 1 + 16]}$
$= \sqrt{18}$

Calculating QR
Q ≡ (1, 6, – 6) and R ≡ (4, 9, – 6)
By using the formula,
Distance QR = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$
So here,
$x_1 = 1, y_1 = 6, z_1 = -6$
$x_2 = 4, y_2 = 9, z_2 = -6$
Distance QR = $\sqrt{[(4 - 1)^2 + (9 - 6)^2 + (-6 - (-6))^2]}$
$= \sqrt{[(3)^2 + (3)^2 + (-6 + 6)^2]}$
$= \sqrt{[9 + 9 + 0]}$
$= \sqrt{18}$

Hence, PQ = QR
18 = 18
2 sides are equal
∴ PQR is an isosceles triangle.

(ii) (0, 7, 10), (–1, 6, 6) and (– 4, 9, 6) are the vertices of a right angled triangle.

Let the points be
P (0, 7, 10), Q (– 1, 6, 6) &R (– 4, 9, 6)
Firstly let us calculate the distance of PQ, OR and PR
Calculating PQ
P ≡ (0, 7, 10) and Q ≡ (– 1, 6, 6)
By using the formula,
Distance PQ= $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$
So here,
$x_1 = 0, y_1 = 7, z_1 = 10$
$x_2 = -1, y_2 = 6, z_2 = 6$
Distance PQ = $\sqrt{[(-1 - 0)^2 + (6 - 7)^2 + (6 - 10)^2]}$
$= \sqrt{[(-1)^2 + (-1)^2 + (-4)^2]}$
$= \sqrt{[1 + 1 + 16]}$
$= \sqrt{18}$

Calculating QR
Q ≡ (1, 6, – 6) and R ≡ (4, 9, – 6)
By using the formula,
Distance QR = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$
So here,
$x_1 = 1, y_1 = 6, z_1 = -6$
$x_2 = 4, y_2 = 9, z_2 = -6$
Distance QR = $\sqrt{[(4 - 1)^2 + (9 - 6)^2 + (6 - (-6))^2]}$
$= \sqrt{[(3)^2 + (3)^2 + (-6 + 6)^2]}$
$= \sqrt{[9 + 9 + 0]}$
$= \sqrt{18}$

Calculating PR
P ≡ (0, 7, 10) and R ≡ (– 4, 9, 6)
By using the formula,
Distance PR = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$
So here,
$x_1 = 0, y_1 = 7, z_1 = 10$
$x_2 = -4, y_2 = 9, z_2 = 6$
Distance PR = $\sqrt{[(-4 - 0)^2 + (9 - 7)^2 + (6 - 10)^2]}$
$= \sqrt{[(-4)^2 + (2)^2 + (-4)^2]}$
$= \sqrt{[16 + 4 + 16]}$
$= \sqrt{36}$
Now,
$PQ^2 + QR^2 = 18 + 18$
$= 36$
$= PR^2$
By using convers of Pythagoras theorem,
∴ The given vertices P, Q& R are the vertices of a right – angled triangle at Q.

(iii) (–1, 2, 1), (1, –2, 5), (4, –7, 8) and (2, –3, 4) are the vertices of a parallelogram.
Let the points be: A (–1, 2, 1), B (1, –2, 5), C (4, –7, 8) &D (2, –3, 4)
ABCD can be vertices of parallelogram only if opposite sides are equal.
i.e. AB = CD and BC = AD
Firstly let us calculate the distance
Calculating AB
A ≡ (– 1, 2, 1) and B ≡ (1, – 2, 5)
By using the formula,
Distance AB = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$
So here,
$x_1 = -1, y_1 = 2, z_1 = 1$
$x_2 = 1, y_2 = -2, z_2 = 5$
Distance AB = $\sqrt{[(1 - (-1))^2 + (-2 - 2)^2 + (5 - 1)^2]}$
$= \sqrt{[(2)^2 + (-4)^2 + (4)^2]}$
$= \sqrt{[4 + 16 + 16]}$
$= \sqrt{36}$
$= 6$
Calculating BC
B ≡ (1, – 2, 5) and C ≡ (4, – 7, 8)
By using the formula,
Distance BC = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$
So here,
$x_1 = 1, y_1 = -2, z_1 = 5$
$x_2 = 4, y_2 = -7, z_2 = 8$
Distance BC = $\sqrt{[(4 - 1)^2 + (-7 - (2))^2 + (8 - 5)^2]}$
$= \sqrt{[(3)^2 + (-5)^2 + (3)^2]}$
$= \sqrt{[9 + 25 + 9]}$

$= \sqrt{43}$

Calculating CD

C ≡ (4, – 7, 8) and D ≡ (2, – 3, 4)

By using the formula,

Distance CD = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

So here,

$x_1 = 4, y_1 = -7, z_1 = 8$

$x_2 = 2, y_2 = -3, z_2 = 4$

Distance CD = $\sqrt{[(2-4)^2 + (-3-(-7))^2 + (4-8)^2]}$

$= \sqrt{[(-2)^2 + (4)^2 + (-4)^2]}$

$= \sqrt{[4 + 16 + 16]}$

$= \sqrt{36}$

= 6

Calculating DA

D ≡ (2, – 3, 4) and A ≡ (– 1, 2, 1)

By using the formula,

Distance DA = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

So here,

$x_1 = 2, y_1 = -3, z_1 = 4$

$x_2 = -1, y_2 = 2, z_2 = 1$

Distance DA = $\sqrt{[(-1-2)^2 + (2-(-3))^2 + (1-4)^2]}$

$= \sqrt{[(-3)^2 + (5)^2 + (-3)^2]}$

$= \sqrt{[9 + 25 + 9]}$

$= \sqrt{43}$

Since AB = CD and BC = DA (given)

So, In ABCD both pairs of opposite sides are equal.

∴ ABCD is a parallelogram.

Question 4

Find the equation of the set of points which are equidistant from the points (1, 2, 3) and (3, 2, –1).

Solution:

Let A (1, 2, 3) & B (3, 2, – 1)

Let point P be (x, y, z)

Since it is given that point P(x, y, z) is equal distance from point A (1, 2, 3) &B (3, 2, – 1) i.e. PA = PB

Firstly let us calculate

Calculating PA

P ≡ (x, y, z) and A ≡ (1, 2, 3)

By using the formula,

Distance PA = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

So here,

$x_1 = x, y_1 = y, z_1 = z$

$x_2 = 1, y_2 = 2, z_2 = 3$
Distance PA= $\sqrt{[(1 - x)^2 + (2 - y)^2 + (3 - z)^2]}$
Calculating PB
P ≡ (x, y, z) and B ≡ (3, 2, – 1)
By using the formula,
Distance PB = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$
So here,
$x_1 = x, y_1 = -y, z_1 = z$
$x_2 = 3, y_2 = 2, z_2 = -1$
Distance PB = $\sqrt{[(3 - x)^2 + (2 - y)^2 + (-1 - z)^2]}$
Since PA = PB
Square on both the sides, we get
$PA^2 = PB^2$
$(1 - x)^2 + (2 - y)^2 + (3 - z)^2 = (3 - x)^2 + (2 - y)^2 + (-1 - z)^2$
$(1 + x^2 - 2x) + (4 + y^2 - 4y) + (9 + z^2 - 6z)$
$(9 + x^2 - 6x) + (4 + y^2 - 4y) + (1 + z^2 + 2z)$
$-2x - 4y - 6z + 14 = -6x - 4y + 2z + 14$
$4x - 8z = 0$
$x - 2z = 0$
∴ The required equation is x – 2z = 0

Question 5

Find the equation of the set of points P, the sum of whose distances from A (4, 0, 0) and B (– 4, 0, 0) is equal to 10.

Solution:

Let A (4, 0, 0) & B (– 4, 0, 0)
Let the coordinates of point P be (x, y, z)
Calculating PA
P ≡ (x, y, z) and A ≡ (4, 0, 0)
By using the formula,
Distance PA = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$
So here,
$x_1 = x, y_1 = y, z_1 = z$
$x_2 = 4, y_2 = 0, z_2 = 0$
Distance PA = $\sqrt{[(4 - x)^2 + (0 - y)^2 + (0 - z)^2]}$
Calculating PB
P ≡ (x, y, z) and B ≡ (– 4, 0, 0)
By using the formula,
Distance PB= $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$
So here,
$x_1 = x, y_1 = y, z_1 = z$
$x_2 = -4, y_2 = 0, z_2 = 0$
Distance PB = $\sqrt{[(-4 - x)^2 + (0 - y)^2 + (0 - z)^2]}$

Now is given that:
PA + P B = 10
PA = 10 – PB
Square on both the sides, we get
$PA^2 = (10 - PB)^2$
$PA^2 = 100 + PB^2 - 20PB$
$(4 - X)^2 + (0 - Y)^2 + 0 - Z)^2$
$100 + (-4 - x)^2 + (0 - y)^2 + (0 - z)^2 - 20PB$
$(16 + x^2 - 8x) + (y)^2 + (z)^2$
$100 + (16 + x^2 + 8x) + (y^2) + (z^2) - 20PB$
20PB = 16x + 100
5PB = (4x +25)
Square on both the sides again, we get
$25PB^2 = 16x^2 + 200x + 625$
$25[(-4 - x)^2 + (0 - y)^2 + (0 - z)^2] = 16x^2 + 200x + 625$
$25[x^2 + y^2 + z^2 + 8x + 16] = 16x^2 + 200x + 625$
$25x^2 + 25y^2 + 25z^2 + 200x + 400 - 16x^2 + 200x + 625$
$9x^2 + 25y^2 + 25z^2 - 225 = 0$
∴ The required equation $9x^2 + 25y^2 + 25z^2 - 225 = 0$

Exercise12.3

Question 1

Find the coordinates of the point which divides the line segment joining the points (– 2, 3, 5) and (1, – 4, 6) in the ratio (i) 2: 3 internally, (ii) 2: 3 externally.

Solution:

Let the line segment joining the points P (-2, 3, 5) and Q (1, -4, 6) be PQ.

(i) 2: 3 internally
By using section formula,
We know that the coordinates of the point R which divides the line segment joining two points P $(x_1, y_1, z_1)$and Q (x_2, y_2, z_2) internally in the ratio m: n is given by:
$$\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}, \frac{mz_2 + nz_1}{m + n}\right)$$
Upon comparing we have
$x_1 = -2, y_1 = 3, z_1 = 5;$
$x_2 = 1, y_2 = -4, z_2 = 6$ and
m = 2, n = 3
So, the coordinates of the point which divides the line segment joining the points P (-2,3, 5) and Q (1, – 4, 6) in the ratio 2: 3 internally is given by:
$$\left(\frac{2 \times 1 + 3 \times (-2)}{2 + 3}, \frac{2 \times (-4) + 3 \times 3}{2 + 3}, \frac{2 \times 6 + 3 \times 5}{2 + 3}\right)$$

$= \left(\frac{2-6}{5}, \frac{-8+9}{5}, \frac{12+15}{5}\right)$

$= \left(\frac{-4}{5}, \frac{1}{5}, \frac{27}{5}\right)$

Hence the coordinates of the point which divides the line segment joining the points (-2, 3, 5) and (1, -4, 6) is (-4/5, 1/5, 27/5)

(ii) 2: 3 externally

By using section formula,

We know that the coordinates of the point R which divides the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) externally in the ratio m: n is given by:

$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$

Upon comparing we have

$x_1 = -2, y_1 = 3, z_1 = 5;$

$x_2 = 1, y_2 = -4, z_2 = 6$ and

m = 2, n = 3

So, the coordinates of the point which divides the line segment joining the points P (– 2, 3, 5) and Q (1, – 4, 6) in the ratio 2: 3 externally is given by:

$\left(\frac{2 \times 1 - 3 \times (-2)}{2-3}, \frac{2 \times (-4) - 3 \times 3}{2-3}, \frac{2 \times 6 - 3 \times 5}{2-3}\right)$

$= \left(\frac{2-(-6)}{-1}, \frac{-8-9}{-1}, \frac{12-15}{-1}\right)$

$= \left(\frac{8}{-1}, \frac{-17}{-1}, \frac{-3}{-1}\right)$

$= (-8, 17\ 3)$

∴ The co-ordinates of the point which divides the line segment joining the points (-2, 3, 5) and (1, -4, 6) is (-8, 17, 3).

Question 2

Given that P (3, 2, – 4), Q (5, 4, – 6) and R (9, 8, –10) are collinear. Find the ratio in which Q divides PR.

Solution:

Let us consider Q divides PR in the ratio k: 1.

By using section formula,

We know that the coordinates of the point R which divides the line segment joining two P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) internally in the ratio m: n is given by:

$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$

Upon comparing we have

$x_1 = 3, y_1 = 2, z_1 = -4;$

$x_2 = 9, y_2 = 8, z_2 = -10$ and

m = k, n = 1

So we have

$$\left(\frac{9k+3}{k+1}, \frac{8k+2}{k+1}, \frac{10k-4}{k+1}\right) = (5,4,-6)$$

$$\frac{9k+3}{k+1} = 5, \frac{8k+2}{k+1} = 4, \frac{10-4}{k+1} = -6$$

9k+3=5(k+1)

9k+3 = 5k + 5

9k – 5k = 5-3

4k = 2

K = 2/4

$= {}^{1}/_{2}$

Hence the ratio in which Q divides PR is 1:2.

Question 3

Find the ratio in which the YZ-plane divides the line segment formed by joining the points (–2, 4, 7) and (3, –5, 8).

Solution:

Let the line segment formed by joining the points P (-2, 4, 7) and Q (3, -5, 8) be PQ.
We know that any point on the YZ-plane is of the form (0, y, z).
So now, let R (0, y, z) divides the line segment PQ in the ratio k: 1.
Then,
Upon comparing we have,
$x_1 = -2, y_1 = 4, z_1 = 7;$
$x_2 = 3, y_2 = -5, z_2 = 8$ and
m = k, n = 1
By using the section formula
We know that the coordinates of the point R which divides the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) internally in the ratio m: n is given by:

$$\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right)$$

So we have,

$$\left(\frac{3k-2}{k+1}, \frac{-5k+4}{k+1}, \frac{8k+7}{k+1}\right) = (0, y, z)$$

$$\frac{3k-2}{k+1} = 0$$

3k -2 = 0

3k = 2

K = 2/3

Hence, the ratio in which the YZ- plane divides the line segment formed by joining the points (–2, 4, 7) and (3, –5, 8) is 2:3.

Question 4

Using section formula, show that the points A (2, –3, 4), B (–1, 2, 1) and C (0, 1/3, 2) are collinear.

Solution:

Let the point P divides AB in the ratio k: 1.
Upon comparing we have,
$x_1 = 2, y_1 = -3, z_1 = 4;$
$x_2 = -1, y_2 = 2, z_2 = 1$ and
m = k, n = 1
By using section formula
We know that the coordinates of the point R which divides the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) internally in the ratio m: n is given by:
$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$
So we have,
The coordinates of P $\left(\frac{-k+1}{k+1}, \frac{2k-3}{k+1}, \frac{k+4}{k+1}\right)$
Now, we check if for some value of k, the point coincides with the point C.
Put (-k+2)/ (k+1) = 0
-k+2 = 0
k = 2
When k = 2, then (2k-3)/ (k+1) = (2(2)-3)/ (2+1)
= (4-3)/3
= 1/3
And, (k+4)/ (k+1) = (2+4)/ (2+1)
= 6/3
= 2
∴ C (0, 1/3, 2) is a point which divides AB in the ratio 2: 1 and is same as P.
Hence, A, B, C are collinear.

Question 5

Find the coordinates of the points which trisect the line segment joining the points P (4, 2, –6) and Q (10, –16, 6).

Solution:

Let A (x_1, y_1, z_1) and B (x_2, y_2, z_2) trisect the line segment joining the points P (4, 2, -6) and Q (10, -16, 6).
A divides the line segment PQ in the ratio 1: 2.
Upon comparing we have,
$x_1 = 4, y_1 = 2, z_1 = -6;$
$x_2 = 10, y_2 = -16, z_2 = 6$ and
m = 1, n = 2
By using section formula
We know that the coordinates of the point R which divides the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) internally in the ratio m: n is given by:

$\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}, \frac{mz_2 + nz_1}{m + n}\right)$
So we have,
The coordinates of A = $\left(\frac{1\times10+2\times4}{1+2}, \frac{1\times(-16)+2\times2}{1+2}, \frac{1\times6+2\times2}{1+2}\right)$
= (18/3, -12/3, -6/3)
= (6, -4, -2)
Similarly, we know that B divides the line segment PQ in the ratio 2: 1.
Upon comparing we have,
$x_1 = 4, y_1 = 2, z_1 = -6;$
$x_2 = 10, y_2 = -16, z_2 = 6$ and
m = 2, n = 2
By using section formula,
We know that the coordinates of the point R which divides the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) internally in the ratio m: n is given by:
$\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}, \frac{mz_2 + nz_1}{m + n}\right)$
So we have,
The coordinates of B= $\left(\frac{2\times10+1\times4}{2+1}, \frac{2\times(-16)+1\times2}{2+1}, \frac{1\times6+1\times(-6)}{2+1}\right)$
= (24/3, -30/3, 6/3)
= (8, -10, 2)
∴ The coordinates of the points which trisect the line segment joining the points P (4, 2, – 6) and Q (10, –16, 6) are (6, -4, -2) and (8, -10, 2).

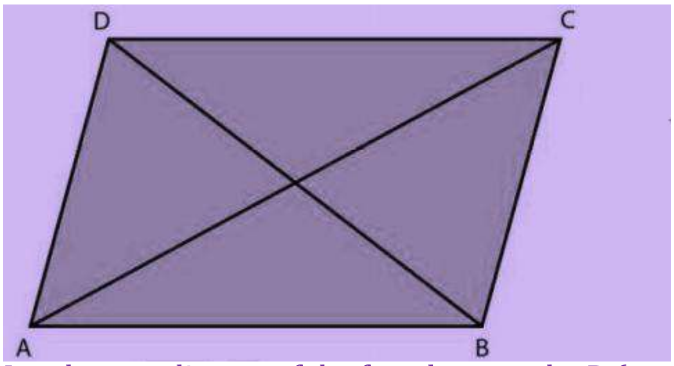
Miscellaneous Exercise

Question 1

Three vertices of a parallelogram ABCD are A (3, – 1, 2), B (1, 2, – 4) and C (– 1, 1, 2). Find the coordinates of the fourth vertex.

Solution:

Given:
ABCD is a parallelogram, with vertices A (3, -1, 2), B (1, 2, -4), C (-1, 1, 2).
Where, $x_1 = 3, y_1 = -1, z_1 = 2;$
$x_2 = 1, y_2 = 2, z_2 = -4;$
$x_3 = -1, y_3 = 2, z_3 = 2$



Let the coordinates of the fourth vertex be D (x, y, z).
We also know that the diagonals of a parallelogram bisect each other, so the mid points of AC and BD are equal, i.e. Midpoint of AC =Midpoint of BD ………. (1)
Now, by midpoint Formula, we know that the coordinates of the mid-point of the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are [(x_1+x_2)/2, (y_1+y_2)/2, (z_1+z_2)/2]
So we have,
Co-ordinates of the midpoint of AC:

$$= \left(\frac{3-1}{2}, \frac{-1+1}{2}, \frac{2+2}{2}\right)$$

= (2/2, 0/2, 4/2)
= (1, 0, 2)
Co- ordinates of the midpoint of BD;

$$= \left(\frac{1+x}{2}, \frac{2+y}{2}, \frac{-4+z}{2}\right)$$

So, using (1), we have

$$= \left(\frac{1+x}{2}, \frac{2+y}{2}, \frac{-4+z}{2}\right) = 0$$

$$\frac{1+x}{2} = 1, \frac{2+y}{2} = 0, \frac{-4+z}{2} = 2$$

1+x = 2, 2+y = 0, -4+z = 2
x = 1, y = -2, z = 8
Hence, the coordinates of the fourth vertex is D (1, -2, 8).

Question 2

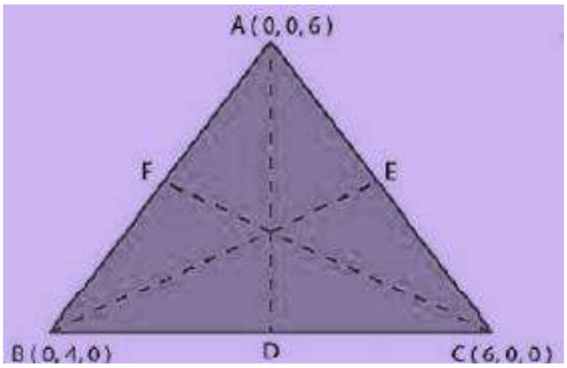
Find the lengths of the medians of the triangle with vertices A (0, 0, 6), B (0, 4, 0) and (6, 0, 0).

Solution:

Given:
The vertices of the triangle are A (0, 0, 6), B (0, 4, 0) and C (6, 0, 0).

$$x_1 = 0, y_1 = 0, z_1 = 6;$$

$x_2 = 0, y_2 = 4, z_2 = 0;$
$x_3 = 6, y_3 = 0, z_3 = 0$



So, let the medians of this triangle be AD, BE and CF corresponding to the vertices A, B and C respectively.
D, E and F are the midpoints of the sides BC, AC and AB respectively.
By midpoint Formula, we know that the coordinates of the mid-point of the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are $[(x_1+x_2)/2, (y_1+y_2)/2, (z_1+z_2)/2]$
So we have,
The coordinates of D:

$$= \left(\frac{0+6}{2}, \frac{4+0}{2}, \frac{0+0}{2}\right) = \left(\frac{6}{2}, \frac{4}{2}, \frac{0}{2}\right)$$
$$= (3, 2, 0)$$

The coordinates of E:

$$= \left(\frac{0+6}{2}, \frac{0+0}{2}, \frac{6+0}{2}\right) = \left(\frac{6}{2}, \frac{0}{2}, \frac{6}{2}\right)$$
$$= (3, 0, 3)$$

And the coordinates of F:

$$= \left(\frac{0+0}{2}, \frac{0+4}{2}, \frac{6+0}{2}\right) = \left(\frac{0}{2}, \frac{4}{2}, \frac{6}{2}\right)$$
$$= (0, 2, 3)$$

By Distance formula, we know that the between two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) is given by
$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
So the lengths of the medians are:

$$AD = \sqrt{(3-0)^2 + (2-0)^2 + (0-6)^2} = \sqrt{3^2 + 2^2 + (-6)^2} = \sqrt{9+4+36}$$
$$= \sqrt{49} = 7$$
$$BE = \sqrt{(3-0)^2 + (0-4)^2 + (3-0)^2} = \sqrt{3^2 + (-4)^2 + 3^2} = \sqrt{9+16+9}$$
$$= \sqrt{34}$$
$$CF = \sqrt{(0-6)^2 + (2-0)^2 + (3-0)^2} = \sqrt{(-6)^2 + 2^2 + (3)^2} = \sqrt{36+4+9}$$
$$= \sqrt{49} = 7$$

∴ The lengths of the medians of the given triangle are 7, √34 and 7.

Question 3

If the origin is the centroid of the triangle PQR with vertices P (2a, 2, 6), Q (– 4, 3b, –10) and R (8, 14, 2c), then find the values of a, b and c.

Solution:

Given:
The vertices of the triangle are P (2a, 2, 6), Q (-4, 3b, -10) and R (8, 14, 2c).
Where,

$$x_1 = 2a, y_1 = 2, z_1 = 6;$$

$x_2 = -4, y_2 = 3b, z_2 = 10;$
$x_3 = 8, y_3 = 14, z_3 = 2c$
We know that the coordinates of the centroid of the triangle, whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3), are $[(x_1+ x_2+ x_3)/3, (y_1+ y_2+ y_3)/3, (z_1+ z_2+ z_3)/3]$
So, the coordinates of the triangle PQR are
$\left(\frac{2a-4+8}{3}, \frac{2+3b+14}{3}, \frac{6-10+2c}{3}\right) = \left(\frac{2a+4}{3}, \frac{3b+16}{3}, \frac{2c-4}{3}\right)$
Now, it is given that the origin (0, 0, 0) is the centroid.
So we have $\left(\frac{2a+4}{3}, \frac{3b+16}{3}, \frac{2c-4}{3}\right) = (0, 0, 0)$
$\left(\frac{2a+4}{3} = 0, \frac{3b+16}{3} = 0, \frac{2c-4}{3} = 0\right)$
2a + 4 = 0, 3b +16 = 0, 2c – 4 = 0
A = -2, b = -16/3, c = 2
∴ The values of a, b and c are a = -2, b = -16/3, c = 2

Question 4

Find the coordinates of a point on y-axis which are at a distance of $5\sqrt{2}$ from the point P (3, –2, 5).

Solution:

Let the point on y-axis be A (0, y, 0).
Then, it is given that the distance between the points A (0, y, 0) and P (3, -2, 5) is 5√2.
Now, by using distance formula,
By Distance formula, we know that the between two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) is given by
Distance of PQ = $\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$
So the distance between the points A (0, y, 0) and P (3, -2, 5) is given by
Distance AP = $\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$
$= \sqrt{[(3-0)^2 + (-2-y)^2 + (5-0)^2]}$
$= \sqrt{[3^2 + (-2-y)^2 + 5^2]}$
$= \sqrt{[(-2-y)^2 + 9 + 25]}$
$5\sqrt{2} = \sqrt{[(-2-y)^2 + 34]}$
Squaring on both the sides, we get
$(-2-y)^2 + 34 = 25 \times 2$
$(-2-y)^2 = 50 - 34$
$4 + y^2 + (2 \times -2 \times -y) = 16$
$y^2 + 4y - 12 = 0$
$y^2 + 6y - 2y - 12 = 0$

y (y+6)-2 (y+6) = 0
(y + 6) (y – 2) = 0
y = -6, y = 2
∴ The points (0, 2, 0) and (0, -6, 0) are the required points on the y-axis.

Question 5

A point R with x-coordinate 4 lies on the line segment joining the points P (2, –3, 4) and Q (8, 0, 10). Find the coordinates of the point R.
[Hint Suppose R divides PQ in the ratio k: 1. The coordinates of the point R are given by $\left(\frac{8k+2}{k+1}, \frac{-3}{k+1}, \frac{10k+4}{k+1}\right)$]

Solution:

Given:
The coordinates of the points P (2, -3, 4) and Q (8, 0, 10).
$x_1 = 2, y_1 = -3, z_1 = 4;$
$x_2 = 8, y_2 = 0, z_2 = 10$
Let the coordinates of the required point be (4, y, z).
So now, let the point R (4, y, z) divides the line segment joining the points P (2, -3, 4) and Q (8, 0, 10) in the ratio k: 1.
By using Section Formula,
We know that the coordinates of the point R which divides the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) internally in the ratio m: n is given by:
$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$
So we have, $\left(\frac{8k+2}{k+1}, \frac{-3}{k+1}, \frac{10k+4}{k+1}\right) = (4, y, z)$
$\Rightarrow \frac{8k+2}{k+1} = 4$
8k + 2 = 4 (k + 1)
8k + 2 = 4k + 4
8k – 4k = 4 -2
4k = 2
k= 2/4
= ½
Now let us substitute the values, we get
$\Rightarrow y = \frac{-3}{\frac{1}{2}+1} = \frac{-3}{\frac{3}{2}} = \frac{-3\times2}{3} = -2$
$$z = \frac{10\left(\frac{1}{2}\right) + 4}{\frac{1}{2} + 1} = \frac{5+4}{\frac{3}{2}} = \frac{9 \times 2}{3} = 3 \times 2$$
= 6
∴ The coordinates of the required point are (4, -2, 6).

Question 6

If A and B the points (3, 4, 5) and (-1, 3, -7), respectively, find the equation of the set of points P such that $PA^2 + PB^2 = k^2$, where k is a constant.

Solution:

Given:
The points A (3, 4, 5) and B (-1, 3, -7)
$x_1 = 3, y_1 = 4, z_1 = 5;$
$x_2 = -1, y_2 = 3, z_2 = -7;$
$PA^2 + PB^2 = k^2$ …………….. (i)
Let the point be (x, y, z)
Now, by using distance formula,
We know that the between two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) is given by
$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
So,
$PA = \sqrt{(3 - x)^2 + (4 - y)^2 + (5 - z)^2}$
And
$PB = \sqrt{(-1 - x)^2 + (3 - y)^2 + (-7 - z)^2}$
Now, substituting these values in (1), we have
$[(3 - x)^2 + (4 - y)^2 + (5 - z)^2] + [(-1 - x)^2 + (3 - y)^2 + (-7 - z)^2] = k^2$
$[(9 + x^2 - 6x) + (16 + y^2 - 8y) + (25 + z^2 - 10z)] + [(1 + x^2 + 2x) + (9 + y^2 - 6y) + (49 + z^2 + 14z)] = k^2$
$9 + x^2 - 6x + 16 + y^2 - 8y + 25 + z^2 - 10z + 1 + x^2 + 2x + 9 + y^2 - 6y + 49 + z^2 + 14z = k^2$
$2x^2 + 2y^2 + 2z^2 - 4x - 14y + 4z + 109 = k^2$
$2(x^2 + y^2 + z^2 - 2x - 7y + 2z) = k^2 - 109$
$(x^2 + y^2 + z^2 - 2x - 7y + 2z) = (k^2 - 109)/2$
Hence, the required equation is $(x^2 + y^2 + z^2 - 2x - 7y + 2z) = (k^2 - 109)/2$

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