Question18

Find the distance of the point (-1,-5,-10) from the point of intersection of the line $\bar{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\bar{r}\,(\hat{i} - \hat{j} + \hat{k}) = 5$

Solution:

Given:
The equation of line is
$\bar{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ …….. (1)
And the equation of the plane is given by
$\bar{r}.(\hat{i} - \hat{j} + \hat{k}) = 5$ ………… (2)
Now to find the intersection of line and plane, substituting the value of $\bar{r}$ from equation (1) of line into equation of plane (2), we get
$[(2\hat{i} - \hat{j} + 2\hat{k}) + \lambda\,(3\hat{i} + 4\hat{j} + 2\hat{k})](\hat{i} - \hat{j} + \hat{k}) = 5$
$[(2 + 3\lambda)\hat{i} + (-1 + 4\lambda)\hat{j} + (2 + 2\lambda)\hat{k}](\hat{i} - \hat{j} + \hat{k}) = 5$
$(2+3\lambda) \times 1 + (-1+4\lambda) \times (-1) + (2+2\lambda) \times 1=5$
$2+ 3\lambda + 1 -4\lambda + 2 + 2\lambda = 2$
So the equation of line is
$\bar{r} = (2\hat{i} - \hat{j} + 2\hat{k})$
Let the point of intersection be (x, y, z)
So,
$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$
$x\hat{i} + y\hat{j} + z\hat{k} = 2\hat{i} - \hat{j} + 2\hat{k}$
Where
X = 2, y = -1, z = 2
So, the point of intersection is (2, -1, 2).
Now, the distance between points (x_1, y_1, z_1) and x_2, y_2, z_2 is given by
$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$ Units
Distance between the points A (2, -1 2) and B (-1, -5, -10) is given by
AB = $\sqrt{(2 - (-1))^2 + (-1 - (-5))^2 + (2 - (-10))^2}$
= $\sqrt{(3)^2 + (4)^2 + (12)^2}$
= $\sqrt{9 + 6 + 144}$
= $\sqrt{169}$
= 13 units
∴ The distance is 13 units.

Question19

Find the vector equation of the line passing through (1, 2, 3) and parallel of the planes $\bar{r}(\hat{i} - \hat{j} + 2\hat{k}) = 5$ and $\bar{r}\,(3\hat{i} + \hat{j} + \hat{k}) = 6$

Solution:

The vector equation of a line passing through a point with position vector $\bar{a}$ and parallel to a vector $\bar{b}$ is
$\bar{r} = \bar{a} + \lambda\bar{b}$

It is given that line passes through (1, 2,3)
So,
$\bar{a} = 1\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$
It is also given that the is line is parallel to both planes.
So line is perpendicular to normal of both planes
i.e. $\bar{b}$ is perpendicular to normal of both planes.
We know that
$\bar{a} \times \bar{b}$ is perpendicular to both $\bar{a}$ and $\bar{b}$
So, $\bar{b}$ is cross product of normal of plane $\bar{r}(\hat{\imath} - \hat{\jmath} + 2\hat{k}) = 5$ and $\bar{r}(3\hat{\imath} - \hat{\jmath} + \hat{k}) = 6$

Required Normal = $\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 2 \\ 3 & 1 & 1 \end{vmatrix}$

= $\hat{\imath}[(-1)(1 - 1(2)] - \hat{\jmath}[1(1) - 3(-1)$
= $\hat{\imath}[-1 - 2] - \hat{\jmath}[1 - 6] + \hat{k}[1 + 3]$
= $-3\hat{\imath} + 5\hat{\jmath} + 4\hat{k}$
So,
$\bar{b} = -3\hat{\imath} + 5\hat{\jmath} + 4\hat{k}$
Now, substitute the value of $\bar{a}$ & $\bar{b}$ in the formula, we get
$\bar{r} = \bar{a} + \lambda\bar{b}$
$-(1\hat{\imath} + 2\hat{\jmath} + 3\hat{k}) + \lambda(-3\hat{\imath} + 5\hat{\jmath} + 4\hat{k})$
∴ The equation of the line is
$\bar{r} = (1\hat{\imath} + 2\hat{\jmath} + 3\hat{k}) + \lambda(-3\hat{\imath} + 5\hat{\jmath} + 4\hat{k})$

Question20

Find the vector equation of the line passing through the point (1, 2, – 4) and perpendicular to the two lines:
$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$ **and** $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$

Solution:

The vector equation of a line passing through a point with position vector $\bar{a}$ and parallel to a vector $\bar{b}$ is $\bar{r} = \bar{a} + \lambda\bar{b}$
It is given that, the line passes through (1, 2, 4)
So,
$\bar{a} = 1\hat{\imath} + 2\hat{\jmath} + 4\hat{k}$
It is also given that; line is parallel to both planes.
We know that
$\bar{a} \times \bar{b}$ is perpendicular to both $\bar{a}$ & $\bar{b}$
So, $\bar{b}$ is cross product of normal planes
$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$

Required Normal = $\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 16 & 7 \\ 3 & 8 & 5 \end{vmatrix}$

= $\hat{\imath}[(-16)(-5) - 8(7)] - \hat{\jmath}\,[3(-5 - 3(7)] + \hat{k}\,[3(8) - 3(-16)]$
= $\hat{\imath}[80 - 56] - \hat{\jmath}[-15 - 21] + \hat{k}[24 + 48]$
= $24\hat{\imath} + 36\hat{\jmath} + 72\hat{k}$

So,

$\bar{b} = 24\hat{i} + 36\hat{j} + 72\hat{k}$

Now, by substituting the value of $\bar{a}$ & $\bar{b}$ in the formula, we get

$\bar{r} = \bar{a} + \lambda\bar{b}$

$= (1\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(24\hat{i} + 36\hat{j} + 72\hat{k}$

$= (= (1\hat{i} + 2\hat{j} - 4\hat{k}) + 12\lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$

$= (1\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$

∴ The equation of the line is

$\bar{r} = (1\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$

Question21

Prove that if a plane has the intercepts a, b, c and is at a distance of p units from the origin, then

$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2}$

Solution:

We know that the distance of the point (x_1, y_1, z_1) from plane Ax +By +Cz =D is given as

$\left|\frac{Ax_1+By_1+Cz_1-D}{\sqrt{A^2+B^2+C^2}}\right|$

The equation of a plane having intercepts a, b, c on the x-, y- z axis respectively is given us

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} -= 1$

Let us compare it with Ax + By +Cz = D, we get

A = 1/a, B = 1/b, C = 1/c, D=1

It is given that, the plane is at a distance of ‘p’ units from the origin.

So, the origin point is o (0, 0, 0,)

Where, $x_1 = 0, y_1 = 0, z_1 = 0$

Now,

Distance = $\left|\frac{Ax_1+By_1+Cz_1-D}{\sqrt{A^2+B^2+C^2}}\right|$

By substituting values in above equation, we get

$$p = \left|\frac{\frac{1}{a}\times0+\frac{1}{b}\times0+\frac{1}{c}\times0-1}{\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2}}\right|$$

$$p = \left|\frac{0+0+0-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\right|$$

$$p = \left|\frac{-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\right|$$

$$p = \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

$$\frac{1}{p} = \sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}$$

Now let us square on both sides, we get

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

Hence proved.