Chapter 6
Application of derivatives
Exercise 6.1

Question 1

Find the rate of change of the area of a circle with respect to its radius r when (a) r = 3 cm (b) r = 4 cm.

Solution:

Consider x denote the area of the circle of radius r.
Area of circle, x = πr^2
And rate of change of area x w.r.t.r is
$\frac{dx}{dr} = \pi(2r) = 2\pi r$
(a) r = 3 cm
$\frac{dx}{dr} = 2\pi\ (3) = 6\pi$ sq. cm
(b) r = 4 cm
$\frac{dx}{dr} = 2\ \pi(4) = 8\ \pi$ sq.cm

Question 2

The volume of a cube is increasing at the rate of 8 cm³/sec. How fast is the surface area increasing when the length of an edge is 12 cm?

Solution:-

Consider a side of the cube be x cm.
Rate of increase of volume of cube = 8 cm^3/sec

- ⇨ $\frac{d}{dt}(xxx) = \frac{d}{dt}(x^3)$
- ⇨ $3x^2\frac{d}{dx}x = 8$
- ⇨ $\frac{d}{dx} = \frac{8}{3x^2}$ ………….. (1)

Consider y be the surface area of the cube, i. e. , y = $6x^2$

Rate of change of surface area of the cube = $\frac{dy}{dt} = 6\frac{d}{dt}x^2$
$= 6\left(2x\frac{dx}{dt}\right) = 12x\left(\frac{8}{3x^2}\right)$
$= 4\left(\frac{8}{x}\right) = \frac{32}{x}$ cm^2/sec
Put x = 12 cm (Given)
$\frac{dy}{dt} = \frac{32}{12} = \frac{8}{3}$ cm^2/sec

As, $\frac{dy}{dt}$ is positive, therefore surface area is increasing at the rate of $\frac{8}{3}$ cm²/sec

Question 3

The radius of the circle is increasing uniformly at the rate of 3 cm per second. Find the rate at which the area of the circle is increasing when the radius is 10 cm.

Solution:-

Consider x cm be the radius of the circle at time t.
Rate of increase of radius of circle = 3 cm/sec.

⇨ $\frac{dx}{dt}$ is positive and equal to 3 cm/sec

Consider y be the area ofthe circle

⇨ $Y = \pi r^2$

∴ Rate of change of area of circle = $\frac{dy}{dt} = \pi\frac{d}{dt}x^2$

$= \pi.\, 2x\frac{dx}{dt} = 2\pi x\ (3)$

$= 6\pi x$

Put x = 10 cm (given),

$\frac{dy}{dt} = 6\pi\ (10) = 60\pi$ cm²/sec

As, $\frac{dy}{dt}$ is positive, therefore surface area is increasing at the rate of 60π cm²/sec.

Question 4

An edge of a variable cube is increasing at the rate of 3 cm per second. How fast is the volume of the cube increasing when the edge if 10 cm long?

Solution:-

Consider x cm be the edge of variable cube at time t.
Rate of increasing of edge = 3 cm/sec.

⇨ $\frac{dx}{dt}$ is positive and = 3 cm/sec

Consider y be the volume of the cube.

⇨ $Y = x^3$

Therefore, Rate of change of volume of cube = $\frac{dy}{dt} = \frac{d}{dt}x^3$

$= 3x^2\frac{dx}{dt} = 3x^2\ (3)$

$= 9x^2$ cm²/sec.

Put x = 10 cm (Given)

$\frac{dy}{dt} = 3(10)^2 = 900\ cm^2/sec$

As, $\frac{dy}{dt} = 3x^2(3)$

$= 9x^2 cm^2/sec$
Put x = 10 cm (given)
$\frac{dy}{dt} = 9(10)^2 = 900\ cm^3/sec$
As, $\frac{dy}{dt}$ is positive, therefore volume of cube is increasing at the rate of $900\ cm^3/sec$.

Question 5

A stone is dropped into a quite lake and waves move in circles at the rate of 5 cm/sec. At the instant when radius of the circular wave is 8 cm, how fast is the enclosed area increasing?

Solution:-

Consider x cm be the radius of the circular wave at the time t.
Rate of increase of radius of the circular wave = 5 cm/sec
$\frac{dx}{dt}$ is positive and = 5 cm/sec
Consider y be the enclosed area of the circular wave.
$Y = \pi x^2$
Rate of change of area $= \frac{dy}{dt} = \pi\frac{d}{dt}x^2$
$= \pi.2x\frac{dx}{dt} = 2\pi x\ (5) = 10\pi x$
Put x = 8 cm (Given)
$\frac{dy}{dt} = 10\pi\ (8) = 80\pi\ cm^2/sec$
As, $\frac{dy}{dt}$ is positive, therefore area of circular wave is increasing at the rate of $80\pi cm^2/sec$.

Question 6

The radius of a circle is increasing at the rate of 0.7 cm/s. what is the rate of its circumference?

Solution:-

Consider x cm be the radius of the circle of time t.
Rate of increase of radius circle = 0.7 cm/sec.
⇨ $\frac{dx}{dt}$ is positive and = 0.7 cm/sec
Consider y be the circumference of the circle.
⇨ $Y = 2\pi x$
Rate of change of circumference of circle $= \frac{dy}{dt}$
$= 2\pi\frac{d}{dt}x = 2\pi(0.7)$
$= 1.4\pi$ cm/sec.

Question 7

The length x of a rectangle is decreasing at the rate of 5 cm/minute. When x = 8 cm and y = 6cm, find the rates of change of (a) the perimeter and (b) the area of the rectangle.

Solution:-

Given: Rate of decrease of length x of rectangle is 5 cm/minute.
$\frac{dx}{dt}$ is negative = -5 cm/minute
Also, Rate of increase of width y of rectangle is 4 cm/minute
⇨ $\frac{dy}{dt}$ is positive
= 4 cm/minute.

(a) Consider z denotes the perimeter of rectangle.
X = 2x + 2y

$$\frac{dz}{dt} = 2\frac{dx}{dt} + 2\frac{dy}{dt}$$

= 2(-5) + 2(4) = -2 is negative.
Therefore, perimeter of the rectangle is decreasing at the rate of 2cm/sec.

(b) Consider z denotes the area of rectangle.

Z = xy

$\frac{dz}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$
= 8(4) + 6(-5) = 2 is positive.
Therefore, Area of the rectangle is increasing at the rate of 2 cm^2/sec.

Question 8

A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimeters of gas per second. Find the rate at which at radius of the balloon increases when the radius is 15 cm.

Solution:-

Consider x cm be the radius of the spherical balloon at time t.
According to the question, $\frac{d}{dt}\left(\frac{4}{3}\pi x^3\right) = 900$

$$\frac{4\pi}{3}\frac{d}{dt}x^3 = 900$$

$$\frac{4\pi}{3}3x^2\frac{dx}{dt} = 900$$

$4\pi x^2 = \frac{dx}{dt} = 900$

$\frac{dx}{dt} = \frac{900}{4\pi x^2}$
$\frac{dx}{dt} = \frac{900}{4\pi(15)^2}$
$\frac{dx}{dt} = \frac{900}{4\pi(225)}$
$\frac{dx}{dt} = \frac{900}{900\pi} = \frac{1}{\pi}$
Radius of balloon is increasing at the rate of $\frac{1}{\pi}$ cm sec.

Question 9

A balloon, which always remains spherical, has a variables radius. Find the rate at which its volume is increasing with the radius when the later is 10 cm.

Solution:-

As we know, volume of sphere, V = $\frac{4}{3}\pi x^3$

$$\frac{dv}{dx} = \frac{d}{dx}\left(\frac{4}{3}\pi x^3\right)$$

$= \frac{4}{3}\pi.\, 3x^2$
$= 4\pi x^2$
$\frac{dv}{dx} = 4\pi(10)^2 = 400\pi$
Therefore, the volume is increasing at the rate of $400\pi cm^3/sec$.

Question 10

A ladder 5 cm long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 cm/s. how fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall?

Solution:-

Consider AB be the ladder and length of ladder is 5 cm. so AB = 5 cm.
Let C is the function of wall and ground, let CA = x meters, CB = y meters

So, according to the equation: As x increases, y decreases
And $\frac{dx}{dt} = 2$ cm/s

In ΔABC,
$AC^2 + BC^2 = AB^2$ [Using Pythagoras theorem]
$x^2 + y^2 = 25$ ……. (1)
$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$2x\,(2) + 2y\frac{dy}{dt} = 0$
$2y\frac{dy}{dt} = -4x$
$\frac{dy}{dt} = \frac{-2x}{y}$ …….. (2)
When x = 4, $16 + y^2 = 25$
Y = 3 [From equation (1)]
From equation (2), $\frac{dy}{dt} = \frac{-2\times4}{3} = \frac{-8}{3}$ cm/s

Question 11

A particle moves along the curve 6y =$x^3 + 2$. Find the points on the curve at which the y-coordinate is changing 8 times as fast as the x-coordinate.

Solution:-

Equation of the curve, 6y = $x^3 + 2$ ………. (1)
Consider (x, y) be the required point on curve (1)

As per the given statements, $\frac{dy}{dx} = 8$ ……. (2)
From equation (1), $6\frac{dy}{dx} = 3x^2$
$6 \times 8 = 3x^2$ [From equation (2)]
$x^2 = \frac{6\times8}{3}$
X = $\pm$ 4 (two values of x)
When x = 4,
6y = 64 + 2
Y = 11
Required point is (4, 11)
When x = -4
6y = -64 + 2
Y = $\frac{-31}{3}$
Required point is $\left(-4, \frac{-31}{3}\right)$

Question 12

The radius of an air bubble is increasing at the rate of ½ cm/s. At what rate is the volume of the bubble increasing when the radius is 1 cm?

Solution:-

As per statement,
$\frac{dx}{dt}$Is positive = $\frac{1}{2}$ cm/sec …….. (1)

Volume of air bubble (z) = $\frac{4\pi}{3}x^3$

⇨ $\frac{dz}{dt} = \frac{4\pi}{3}\frac{d}{dt}x^3$

$= \frac{4\pi}{3}3x^2\frac{dx}{dt}$

$= 4\pi x^2\left(\frac{1}{2}\right)$

⇨ $\frac{dz}{dt} = 2\pi x^2$

$= 2\pi(1)^2 = 2\pi$

Therefore, required rate of increase of volume of air bubble is $2\pi cm^3/sec$.

Question 13

A balloon which always remains spherical has a variable diameter $\frac{3}{2}(2x+1)$ find the rate of change of its volume with respect to x.

Solution:-

Given: Diameter of the balloon = $\frac{3}{2}(2x+1)$

And, Radius of the balloon = $\frac{3}{4}(2x+1)$

So, volume of the balloon = $\frac{4}{3}\pi\left(\frac{3}{4}(2x+1)\right)^3$

$= \frac{9\pi}{16}(2x+1)^3$ cubic units

Now, Rate of change of volume w.r.t. x = $\frac{dv}{dx}$

$= \frac{9\pi}{16}3(2x+1)^2 . \frac{d}{dx}(2x+x)$

$= \frac{27\pi}{16}(2x+1)^2 . 2$

$= \frac{27\pi}{8}(2x+1)^2$

Question 14

Sand is pouring from a pipe at the rate of $12cm^3/s$. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

Solution:-

Consider the height and radius of the sand-cone formed at time t second be y cm and x cm respectively.

As per the given statement, y = $\frac{1}{6}x$

⇨ X = 6y

Volume of cone (V) = $\frac{1}{3}\pi x^2 y$

$= \frac{1}{3}\pi(6y)^2 y$

$= 12\pi y^3$

$\Rightarrow \frac{dv}{dy} = 36\pi y^2$

Now, As, $\frac{dv}{dt} = 12$

$\Rightarrow \frac{dv}{dy} \times \frac{dy}{dt} = 12$

$\Rightarrow 36\pi y^2 \times \frac{dy}{dt} = 12$

$\Rightarrow \frac{dy}{dt} = \frac{1}{3\pi y^2}$

$\Rightarrow \frac{dy}{dt} = \frac{1}{3\pi 4^2} = \frac{1}{48\pi}$ cm/sec

Question 15

The total cost C(x) in rupees associated with the production of x units of an item given by c(x) = $0.007\,x^3 - 0.003x^2 + 15x + 4000$.
Find the marginal cost when 17 units are produced.

Solution:-

Marginal cost = $\frac{dc}{dx}$

$= \frac{d}{dx}(0.007\,x^3 - 0.003x^2 + 15x + 4000)$

$= 0.021x^2 - 0.006x + 15$

Now, when x = 17, MC is

$= 0.021(17)^2 - 0.006\times17+15$

= 6.069 – 0.102 + 15 = 20.967

Therefore, required Marginal cost is Rs. 20.97

Question 16

The total revenue in rupees received from the sale of x units of a product is given by R(x) = $13x^2+26x + 15$.
Find the marginal revenue when x = 7.

Solution:-

Marginal Revenue (MR) = $\frac{dR}{dx}$

$= \frac{d}{dx}(13x^2 + 26x + 15)$

= 26x+26

Now, when x = 7, MR is

$= 26 \times 7 + 26 = 208$

Therefore, the required marginal revenue is Rs. 208.

Choose the correct answer in Exercise 17 and 18.

Question 17

The rate of change of the area of a circle with respect to its radius r at r = 6 cm is:

a) 10π
b) 12π
c) 8π
d) 11π

Solution:-

Option (B) is correct.

Area of circle (A) = πr^2

$\Rightarrow \frac{dA}{dr} = 2\pi r$

$= 2\pi \times 6 = 12\pi$

Question 18

The total revenue in Rupees received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. The marginal revenue, when x = 15 is:

a) 116
b) 96
c) 90
d) 126

Solution:-

Option (D) is correct

Total revenue $R(x) = 3x^2 + 36x + 5$

Marginal revenue = $\frac{d}{dx}R(x) = 6x + 36 = 6 \times 15 + 36 = 126$

Exercise 6.2

Question 1

Show that the function given by f(X) = 3x+17 is strictly increasing on R.

Solution:-

Given function: f(x) = 3x+17
Derivate w.r.t.x:
F'(x) = 3(1) +0= 3>0 that is, positive for all x ∈ R
Therefore, f(x) is strictly increasing on R.

Question 2

Show that the function given by f(x) = e^{2x} is strictly increasing on R.

Solution:-

Given function: f(x) = e^{2x}
F'(x) = $e^{2x}\frac{d}{dx}$ 2x = e^{2x} (2) = $2e^{2x}$>0 that is, positive for all x ∈ R
Therefore, f(x) is strictly increasing on R.

Question 3

Show that the function given by f(x) = sin x is

a) Strictly increasing $\left(0, \frac{\pi}{2}\right)$.
b) Strictly decreasing in $\left(\frac{\pi}{2}, \pi\right)$
c) Neither increasing nor decreasing (0, π).

Solution:-

Given function f (x) = sin x
F'(x) = cos x

(a) Since, f'(x) = cos x > 0, that is, positive in 1st quadrant, that is, in $\left(0, \frac{\pi}{2}\right)$.
Therefore, f(x) is strictly increasing in $\left(0, \frac{\pi}{2}\right)$.

(b) Since, f'(x) = cos x < 0, that is, negative in 2nd quadrant, that is, in $\left(\frac{\pi}{2}, \pi\right)$

F(x) is strictly decreasing in $\left(\frac{\pi}{2}, \pi\right)$.
(c) Since f'(x) = cos x > 0, that is positive in 1st quadrant, that is, in $\left(0, \frac{\pi}{2}\right)$ and f'(x) = cos x < 0, that is, negative in 2nd quadrant, that is, in $\left(\frac{\pi}{2}, \pi\right)$ and f' $\left(\frac{\pi}{2}\right)$ = cos$\frac{\pi}{2}$ = 0
Therefore, f'(x) is neither increasing nor decreasing in (0, π)
So, f(x) is neither increasing nor decreasing in (0, π).

Question 4

Find the integrals in which the function f given by f(x) = $2x^2 - 3x$ is

a) Strictly increasing
b) Strictly decreasing

Solution:-

Given function: f(x) = $2x^2 - 3x$
f'(x) = 4x – 3 ……… (1)
Now 4x-3 = 0
⇨ X = $\frac{3}{4}$
Therefore, we have two interviews $\left(-\infty, \frac{3}{4}\right)and\left(\frac{3}{4}, \infty\right)$.
(a) For interval$\left(\frac{3}{4}, \infty\right)$, picking x = 1, then from equation (1), f'(x) > 0.
Therefore, f is strictly increasing in$\left(\frac{3}{4}, \infty\right)$.
(b) For interval$\left(-\infty, \frac{3}{4}\right)$, picking x=0.5, then from equation (1), f'(x) < 0.
Therefore, f is strictly decreasing in$\left(-\infty, \frac{3}{4}\right)$.

Question 5

Find the intervals in which the function f given by f(x) = $2x^3 - 3x^2 - 36x + 7$ is

a) Strictly increasing
b) Strictly decreasing

Solution:-

(a) Given function: f(x) = $2x^3 - 3x^2 - 36x + 7$
F'(x) = $6x^2 - 6x + 36 = 6(x^2 - x - 6)$
Now 6(x+2)(x - 3) = 0
X +2 = 0 or x – 3 = 0
X = -2 or x = 3
The value of x is either -2 or 3.
Therefore, we have sub-intervals are (-x, -2), (-2, 3) and (3, x).
For interval (-2, 3), picking x = 2, from equation (1).
F'(x) = (+)(+)(-)=(-) <0
Therefore, f is strictly increasing in (-x, -2).
For interval ((-2, 3), picking x = 2, from equation (1).
F'(x) = (+)(+)(-) = (-) < 0
Therefore, f is strictly decreasing in (-2, 3)

For interval (3,∞), picking x = 4, from equation (1),
F'(x) = (+)(+)(+) = (+)>0
Therefore, f is strictly increasing in (3,∞).
So, (a) f is strictly increasing in (-∞, -2) and (3,∞).

(b) f is strictly decreasing in (-2, 3).

Question 6

Find the intervals in which the following functions are strictly increasing or decreasing:

a) $x^2 + 2x - 5$
b) 10 - 6x - $2x^2$
c) $2x^3 - 9x^2 - 12x + 1$
d) 6 – 9x - x^2
e) $(x+1)^3(x-3)^3$

Solution:-

(a) Given function: f(x) = x^2+2x-5
➔ f'(x) = 2x+2 = 2(x + 1) ……… (1)
Now 2(x + 1) = 0
➔x = -1
Therefore, we have two sub – intervals (-∞, -1) and (-1,∞).
For interval (-∞, -1) picking x = -2, from equation (1), f'(x) = (-) < 0
Therefore, f is strictly decreasing.
For interval (-1,∞) picking x = -2, from equation (1), f'(x) = (+) > 0
Therefore, f is strictly increasing.

(b) Given function: f(x) = 10 – 6x - $2x^2$
➔ f'(x) = -6-4x = -2(3 + 2x) ……. (1)
Now -2(3x+2x) = 0
X = $\frac{-3}{2}$
Therefore, we have two sub-intervals $\left(-\infty, \frac{-3}{2}\right)$ and$\left(\frac{-3}{2}, \infty\right)$.
For interval $\left(-\infty, \frac{-3}{2}\right)$ picking x = -2, from equation (1),
F'(x) = (-) (-) = (+) > 0
Therefore, f is strictly increasing.
For interval $\left(\frac{-3}{2}, \infty\right)$ picking x = -1, from equation (1),
f'(x) = (-) (+) = (-) <0
Therefore, f is strictly decreasing.

(c) Given function: f(x) = -2x^3 - 9x^2 $- 12x + 1$
Derivate w.r.t. x,

F'(x) = $-6x^2 - 18x - 12$
F'(x) = -6($x^2 + 3x + 2$)
= -6(x+1)(x+2) ..... (1)
Now,
-6(x+1)(X+2) = 0
➔ x= -1 or x = -2
Therefore, we have three disjoint intervals (-∞, -2), (-2, -1) and (-1, ∞).
For
Interval$(-\infty, -2)$, from equation (1),
F'(x) = (-)(-)(-) = (-) < 0
Therefore, f is strictly decreasing.
For interval (-2, -1), from equation (1).
F'(x) = (-) (-) (+) = (+) > 0
Therefore, f is strictly increasing.
For interval (-1, ∞), from equation (1)
F'(x) = (-) (+)(+) = (-) < 0
Therefore, f is strictly decreasing.

(d) Given function: f(x) = 6 – 9x - x^2
F'(x) = - 9 - 2x
Now – 9 - 2x = 0
X = $\frac{-9}{2}$
Therefore, we have three disjoint intervals $\left(-\infty, \frac{-9}{2}\right)$ and$\left(\frac{-9}{2}, \infty\right)$.
For intervals$\left(-\infty, \frac{-9}{2}\right)$. x<$\frac{-9}{2}$
Therefore, f is strictly increasing.
For interval$\left(\frac{-9}{2}, \infty\right)$, x>$\frac{-9}{2}$
Therefore, f is strictly decreasing.

(e) Given function: f (x) = $(x+1)^3(x-3)^3$
F'(x) = $(x+1)^3 . 3(x-3)^2$ + $(x-3)^3$.$3(x+1)^2$
F'(x) = $3(X+1)^2(x-3)^2$(X+1+X - 3)
F'(x) = $3(x+1)^2(x-3)^2$ (2x – 2)
F'(x) = $6(x+1)^2(x-3)^2$(x – 1)
Here, factors $(x+1)^2$ and $(x-3)^2$ are non-negative for all x.
Therefore, f(x) is strictly increasing if f'(x)>0.
x-1>0
x>1
So, f is strictly increasing in (1, ∞) and f is strictly decreasing in (-∞, 1)

Question 7

Show that y = log (1+x)-$\frac{2x}{2+x}$, x>-1 is an increasing function of x throughout its domain.

Solution:-

Given function: $y = \log(1+x) - \frac{2x}{2+x}$

Derivate y w.r.t.x. we have

$\frac{dy}{dx} = \frac{1}{1+x}\frac{d}{dx}(1+x) - \left[\frac{(2+x)\frac{d}{dx}(2x)-2c\frac{d}{dx}(2+x)}{(2+x)^2}\right]$

$= \frac{1}{1+x} - \left[\frac{(2+x)2-2x}{(2+x)^2}\right]$

$= \frac{1}{1+x} - \frac{4}{(2+x)^2}$

This implies,

$\frac{dy}{dx} = \frac{(2+x)^2-4(1+x)}{(1+x)(2+x)}$

$= \frac{x^2}{(1+x)(2+x)^2}$ ……….. (1)

Domain of the given function is given to be x > -1

⇨ x + 1 > 0

Also $(2+x)^2 > 0$ and $x^2 \geq 0$

From equation (1), $\frac{dy}{dx} \geq 0$ for all x in domain x > -1 and f is an increasing function.

Question 8

Find the value of x for which $y = \{x(x-2)\}^2$ is an increasing function.

Solution:-

Given function: $f(x) = y = (x(x-2))^2$

Derivate y w.r.t. x, we get

⇨ $\frac{dy}{dx} = 2x(x-2)\frac{d}{dx}[x(x-2)]$

⇨ $\frac{dy}{dx} = 2x(x-2)\left[x\frac{d}{dx}(x-2) + (X-2)\frac{d}{dx}x\right]$

[Applying product Rule]

⇨ $\frac{dy}{dx} = 2x(x-2)[x+x-2]$

⇨ $2x(x-2)(2x-2)$

$= 4x(x-2)(x-1)$ ……. (1)

⇨ x = 0, x = 2, x = 1

Therefore, we have (-∞, 0), (0, 1), (1, 2), (2, ∞)

For (-∞, 0) picking x = -1

$\frac{dy}{dx} = (-)(-) = (-) \leq 0$

∴ f (x) is decreasing.

For (0, 1) picking $x = \frac{1}{2}$,

$\frac{dy}{dx} = (+)(-)(-) = (+) \geq 0$

$\therefore$ f (x) is increasing.
For (1, 2) Picking x = 1.5,
$\frac{dy}{dx}$ = (+)n (-)(+)= (-) ≤ 0
f(x)is decreasing
For $(2,\infty)$ picking x = 3,
$\frac{dy}{dx}$ =(+)(+)(+) = (+)≥ 0
f(x) is increasing.

Question 9

Prove that y = $\frac{4 \sin\theta}{(2+\cos\theta)} = \theta$is an increasing function of θ in $\left[0, \frac{\pi}{2}\right]$

Solution:-

Given function:
Derivate y w.r.t.θ,

$$\frac{dy}{d\theta} = \frac{(2+\cos\ \theta).4\cos\ \theta - 4\sin\ \theta\ (-\sin\ \theta)}{(2+\cos\ \theta)^2} - 1$$

$$= \frac{8\cos\ \theta + 4\cos^2\theta + 4\sin^2\theta}{(2+\sin\ \theta)^2} - 1$$

$$\frac{dy}{d\theta} = \frac{8\cos\ \theta + 4\left(\cos^2\theta + \sin^2\theta\right) - (2+\cos\ \theta)^2}{(2+\cos\ \theta)^2}$$

$$= \frac{8\cos\ \theta + 4 - (2+\cos\ \theta)^2}{(2+\cos\ \theta)^2}$$

$$\Rightarrow \frac{dy}{d\theta} = \frac{(8\cos\ \theta + 4) - (4 + 4\cos\ \theta + \cos^2\theta)}{(2+\cos\ \theta)^2}$$

$$= \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}$$

Since $0 \leq \theta \leq \frac{\pi}{2}$ and we have $0 \leq \cos\theta \leq 1$, therefore $4 - \cos\theta > 0$.

$\frac{dy}{d\theta} \geq 0$ for $0 \leq \theta \leq \frac{\pi}{2}$

So, y is an increasing function of θin$\left[\theta, \frac{\pi}{2}\right]$.

Question 10

Prove that the logarithmic function is strictly increasing on $(0, \infty)$.

Solution:-

Given function: f (x) = log x
F'(x) = $\frac{1}{x}$ for all x in $(0,\infty)$.
Therefore, f(x) is strictly increasing on $(0,\infty)$.

Question 11

Prove that the function f given by f (x) = $x^2 - x + 1$ is neither strictly increasing nor strictly decreasing on (-1, 1).

Solution:-

Given function: f (x) = $x^2 - x + 1$
F' (x) = 2x – 1
F(x) is strictly increasing if f'(x) > 0
2x – 1 > 0
$x > \frac{1}{2}$
That is, increasing on the interval $\left(\frac{1}{2}, 1\right)$
F(x) is strictly decreasing if f'(x) <0
2x – 1 < 0
$x > \frac{1}{2}$
That is, decreasing
On the interval $\left(-1, \frac{1}{2}\right)$
So, f(x) is neither strictly increasing nor decreasing on the interval (-1, 1).
Question 12

Which of the following functions are strictly decreasing on $\left(0, \frac{\pi}{2}\right)$?

a) Cos x
b) Cos 2x
c) Cos 3x
d) Tan x

Solution:-

(A) Let f(x) = cos x
F'(x) = - sin x
Since $0<x<\frac{\pi}{2}$in$\left(0, \frac{\pi}{2}\right)$, therefore sin x>0
⇨ - sin x < 0
Therefore, f(x) is strictly decreasing on$\left(0, \frac{\pi}{2}\right)$.

(B) f (x) = cos 2x
F'(x) = -2 sin 2x
Since $0<x<\frac{\pi}{2}$
0<2x<π therefore sin 2x > 0
⇨ -2sin 2x<0
Therefore, f(x) is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$

(C) $F(x) = \cos 3x$
$F'(x) = -3 \sin 3x$
Since $0<x<\frac{\pi}{2}$
$0<3x<\frac{3\pi}{2}$
For $0 < 3x < \pi$ $\sin 3x > 0$
$\Rightarrow -3 \sin 3x < 0$
Therefore, f(x) is strictly decreasing on $\left(0, \frac{\pi}{3}\right)$.
So, f(x) is neither strictly increasing not strictly decreasing on $\left(0, \frac{\pi}{2}\right)$.

(D) Let $f(x) = \tan x$
$F'(x) = \sec^2 x > 0$
Therefore, f(x) is strictly increasing on $\left(0, \frac{\pi}{2}\right)$.

Question 13

On which of he following intervals is the function f given by f(x) =x^{100} + sin x – 1 is strictly decreasing:

a) $(0, 1)$
b) $\left(\frac{\pi}{2}, \pi\right)$
c) $\left(0, \frac{\pi}{2}\right)$
d) None of these

Solution:-

Given function: $f(x) = x^{100} + \sin x - 1$
$F'(x) = 100x^{99} + \cos x$

(A) On (0, 1), x>0 therefore $100x^{99} > 0$
And for cos x
(0, 1 radian) = $(0, 57^0\ nearly) > 0$
Therefore, f(x) is strictly increasing on (0, 1).

(B) For $100x^{99}$ $x \in \left(\frac{\pi}{2}, \pi\right)$
For interval: $\left(\frac{11}{7}, \frac{22}{7}\right)$ = (1.5, 3.1) > 1 and so, cos x is negative and between -1 and 0.
Therefore, f(x) is strictly increasing on $\left(\frac{\pi}{2}, \pi\right)$

(c) $\left(0, \frac{\pi}{2}\right)$ = (0, 1.5) both terms of given function are positive.

Therefore, f (x) is strictly increasing on $\left(0, \frac{\pi}{2}\right)$

(D) Option (D)is the correct answer.

Question 14

Find the least value of ''a'' such that the function f. given by f(x) = $x^2 + ax + 1$ strictly increasing on (1, 2).

Solution:-

F(x) = $x^2 + ax + 1$
Apply derivative:
F'(x) = 2x + a

Since f(x) is strictly increasing on (1, 2), therefore f'(x) = 2x + a > 0 for all x in (1, 2) on (1, 2) 1 < x < 2
2<2x<4
2 + a < 2x + a < 4 + a
Therefore, Minimum value of f'(x) is 2 + a and maximum value is 4 + a.
Since f'(x) > 0 for all x in (1, 2)
2 + a >0 and 4 + a > 0
A > -2 and a > -4
Therefore, least value of a is -2.

Question 15

Let I be any interval disjoint from [-1, 1] prove that the function f given by f (x) = x + $\frac{1}{x}$
Is strictly increasing on I.

Solution:-

Given function:
F(x) = x + $\frac{1}{x}$ = x + x^{-1}
Apply derivative:
F'(x) = 1 + (-1)x^{-2} = 1 - $\frac{1}{x^2}$ = $\frac{x^2-1}{x^2}$
F'(x) = $\frac{(x-1)(x+1)}{x^2}$ ………….. (1)
For every x either x < -1 or x > 1
For x < -1, x = -2,
F'(x) = $\frac{(-)(-)}{(+)}$ = (+) > 0
Again for, x > 1, x = 2,

$F'(x) = \frac{(+)(+)}{(+)} = (+) > 0$

$F'(x) > 0$ for all $x \in I\ (-\infty, \infty)$, so f(x) is strictly increasing on I.

Question 16

Prove that the function f given by f(x) = log sin x is strictly increasing on $\left(0, \frac{\pi}{2}\right)$ and strictly decreasing on $\left(\frac{\pi}{2}, \pi\right)$.

Solution:-

Given function:
F(x) = log sin x

Apply derivative:
$F'(x) = \frac{1}{\sin x}\frac{d}{dx}\sin x = \frac{1}{\sin x}\cos x = \cot x$

On the interval $\left(0, \frac{\pi}{2}\right)$ that is, in 1st quadrant,

$F'(x) = \cot x > 0$

Therefore, f(x) is strictly increasing on $\left(0, \frac{\pi}{2}\right)$

On the interval $\left(\frac{\pi}{2}, \pi\right)$ that is, in 2nd quadrant,

$F'(x) = \cot x < 0$

Therefore, f (x) is strictly deceasing on $\left(\frac{\pi}{2}, \pi\right)$.

Question 17

Prove that the function f given by f(x) = log cos x is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$ and strictly decreasing on $\left(\frac{\pi}{2}, \pi\right)$.

Solution:-

Given function: f(x) = log cos x

⇨ $\frac{1}{\cos x}\frac{d}{dx}\cos x = \frac{1}{\cos x}(-\sin x) = -\tan x$

On the interval $\left(0, \frac{\pi}{2}\right)$,

In 1st quadrant, tan x is positive,

Thus $f'(x) = -\tan x < 0$

Therefore, f(x) is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$.

On the interval $\left(\frac{\pi}{2}, \pi\right)$,
In 2nd quadrant, tan x is negative
Thus f'(x) = - tan x > 0
Therefore, f(x) is strictly increasing on $\left(\frac{\pi}{2}, \pi\right)$

Question 18

Prove that the function given by $f(x) = x^3 - 3x^3 + 3x - 100$ is increasing in R.

Solution:-

Given function:
$F(x) = x^3 - 3x^3 + 3x - 100$
Apply derivate:
$F'(x) = 3x^3 - 6x + 3 = 3(x^2 - 2x + 1)$
$F'(x) = 3(x - 1)^2 \geq 0$ for all x in R.
Therefore, f(x) is increasing on R.

Question 19
The interval in which $y = x^2e^{-x}$ is increasing in:

a) $(-\infty, \infty)$
b) $(-2, 0)$
c) $(2, \infty)$
d) $(0, 2)$

Solution:

Option (d)
Explanation:

Given function;
$y = x^2e^{-x}$
Apply derivate:
$\frac{dy}{dx} = x^2 \frac{d}{dx} e^{-x} + e^{-x} \frac{d}{dx} x^2$
$= x^2e^{-x}(-1) + e^{-z}(2x)$
⇨ $\frac{dy}{dx} = -x^2e^{-z} + 2xe^{-x}$
$= xe^{-x} (-x + 2)$
So, $\frac{dy}{dx} = \frac{x(2-x)}{e^x}$
In option (D), $\frac{dy}{dx} > 0$ for all x in the interval (0, 2).

Exercise 6.3

Question 1

Find the slope of tangent to the curve y = $3x^4 - 4x$ at x = 4.

Solution:-

Equation of the curve y = $3x^4 - 4x$ ..... (1)

Slope of the tangent to the curve = value of $\frac{dy}{dx}$ at the point (x, y).

$\frac{dy}{dx}$= $3(4x^3)$ - 4 = $12x^3$- 4

Slope of the tangent at point x = 4 to the curve (1)

= $12(4)^3$ - 4 = 764

Question 2

Find the slope of tangent to the curve y =$\frac{x-1}{x-2}$, x ≠ 2 at x = 10.

Solution:-

Equation of the curve y = $\frac{x-1}{x-2}$ .... (1)

Derivate y w.r.t. x,

$\frac{dy}{dx} = \frac{(x-2)\frac{d}{dx}(x-1)-(x-1)\frac{d}{dx}(x-2)}{(x-2)^2}$

= $\frac{(x-2)-(x-1)}{(x-2)^2}$

= $\frac{-1}{(x-2)^2}$ ....(2)

Slope of the tangent at point x = 10 to the curve (1)

= $\frac{-1}{(10-2)^2}$

= $\frac{-1}{8^2} = \frac{-1}{64}$

Question 3

Find the slope of tangent to the curve y = $x^3 - x + 1$ at the given point whose x – coordinate is 2.

Solution:-

Equation of the curve y = x^3 - x + 1 …….. (1)

Apply derivate w.r.t.x

$\frac{dy}{dx} = 3x^2 - 1$

Slope of the tangent of point x = 2 to the curve (1)

= $3(2)^2$-1 = 11

Question 4

Find the slope of tangent to the curve y = $x^3 - 3x + 2$ at the given point whose x-coordinate is 3.

Solution:-

Equation of the curve y = x^3 - 3x + 2 ……. (1)

Apply derivate w.r.t.x,

$\frac{dy}{dx} = 3x^2 - 3$

Slope of the tangent at the point x = 3 to the curve (1)

= $^3(3)^2$-3 = 24

Question 5

Find the slope of the normal to the curve x = $cos^3\theta, y = a\ sin^3\theta$ at θ = $\frac{\pi}{4}$

Solution:-

Equation of the curve are x = $acos^3\theta$, y = a $sin^3\theta$

x = $acos^3\theta$,

Apply derivate w.r.t.x,

$\frac{dx}{d\theta} = a\frac{d}{d\theta}(\cos\theta)^3$

$= a.3(\cos\theta)^2\frac{d}{d\theta}(\cos\theta)$

⇨ $\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$ …….. (1)

And,

$Y = a\sin^2\theta$

Apply derivate w.r.t.x,

$\frac{dy}{d\theta} = a\frac{d}{d\theta}(\sin\theta)^3$

$a.3(\sin\theta)^2\frac{d}{d\theta}(\sin\theta)$

$\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$ …. (2)

Using (1) and (2), we have

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}$$

$= \frac{-\sin\theta}{\cos\theta} = -\tan\theta$

Now,

Slope of the tangent at $\theta = \frac{\pi}{4}$

$= -\tan\frac{\pi}{4} = -1$

And slope of the normal at $\theta = \frac{\pi}{4}$

$= \frac{-1}{m} = \frac{-1}{-1} = 1$

Question 6

Find the slope of the normal to the curve $x = 1 - a\sin\theta$, $y = b\cos^2\theta$ at $\theta = \frac{\pi}{2}$

Solution:-

Equation of the curves are x = 1 – a sin θ and y = b $\cos^2\theta$

X = 1 – a sin θ

Apply derivative w.r.t.x, we have

$\frac{dx}{d\theta} = 0 - a\cos\theta$

⇨ $\frac{dx}{d\theta} = -a\cos\theta$

Again,

Y = b $\cos^2\theta$

$$\frac{dy}{d\theta} = b\frac{d}{d\theta}(\cos\theta)^2$$

$$\frac{dy}{d\theta} = b.2\cos\theta\frac{d}{d\theta}\cos\theta = 2b\cos\theta\sin\theta$$

Apply derivate w.r.t.x, we have

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-2b\cos\theta\sin\theta}{-a\cos\theta}$

$= \frac{2b}{a}\sin\theta$

Again, slope of the tangent at $\theta = \frac{\pi}{2}$

$= \frac{2b}{a}\sin\frac{\pi}{2} = \frac{2b}{a}$

And slope of the normal at $\theta = \frac{\pi}{2}$

$= \frac{-1}{m} = \frac{-1}{2b/a}$

$= \frac{-a}{2b}$

Question 7

Find the point at which the tangent to the curve y = $x^3 - 3x^3 - 9x + 7$ is parallel to the x-axis.

Solution:-

Equation of the curve y = $x^3 - 3x^2 - 9x + 7$ .... (1)

$\frac{dy}{dx} = 3x^2 - 6x - 9$

Since, the tangent is parallel to the x – axis, so, $\frac{dy}{dx} = 0$

$$3x^2 - 6x - 9 = 0$$

$$x^3 - 2x - 3 = 0$$

(x – 3) (x + 1) = 0

x = 3, x = -1

From equation (1), when x = 3.

Y = 27 – 27 + 7 = -20

When x = -1, y = -1-3+9+7 = 12

Therefore, the required points are (3, -20) and (-1, 12).

Question 8

Find the point on the curve y = $(x - 2)^2$ at which the tangent is parallel to the chord joining the points (2, 0) and (4, 4).

Solution:-

Let the given points are M (2, 0) and N (4, 4)

Slope of the chord, MN = $\frac{4-0}{4-2} = 2$

$$\left[\because m = \frac{y_2 - y_1}{x_2 - x_1}\right]$$

Equation of the curve is y = $(x - 2)^2$ (Given)

Slope of the tangent at (x, y)

$= \frac{dy}{dx} = 2\,(x - 2)$

If the tangent is parallel to the chord MN, then

Slope of tangent = slope of chord

2 (x – 2) = 2

X = 3

Therefore, y = $(3-2)^2$ = 1

Therefore, the required point is (3, 1).

Question 9

Find the point on the curve y = $x^3 - 11x + 5$ at which the tangent is y = x – 11.

Solution:-

Equation of the curve y = x^3 - 11x + 5 …… (1)

Equation of the tangent y = x – 11 ….. (2)

⇨ x – y – 11 = 0

Slope of the tangent at (x, y)

= $\frac{dy}{dx} = 3x^2 - 11$ [From equation (1)]

Slope of tangent = $\frac{-a}{b}$ = $\frac{-1}{-1}$ = 1

[From equation (2)]

Therefore,

$$3x^2 - 11 = 1$$

x^2 = 4

X = ± 2

From equation (1), when x = 2, y = 8 – 22 + 5 = -9

And when x = -2, y = -8 + 22 + 5 = 19

We observed that, (-2, 19) does not satisfy equation (2), therefore the required point is (2, -9).

Question 10

Find the equation of all lines having slope – 1 that are tangents to the curve y = $\frac{1}{x-1}$, x ≠1.

Solution:

Equation of the curve y = $\frac{1}{x-1}$ = $(x-1)^{-1}$ ……… (1)

$\frac{dy}{dx}$ = (-1)$(x-1)^{-2}\frac{d}{dx}$ (x – 1)

= $\frac{-1}{(x-1)^2}$ = Slope of the tangent at (x, y)

But according to given statement, slope = -1

$\frac{-1}{(x-1)^2}$ = -1

$(x-1)^2$ = 1

x – 1 = $\pm$ 1

x = 1 + 1 = 2 or x = 1 – 1 = 0

From equation (1), when x = 2

Y = $\frac{1}{2-1} = -1$

And when x = 0

Y = $\frac{1}{0-1}$ = -1

Points of contact are (2, 1) and (0, -1).

And equation of two tangents are y – 1 = -1 (x – 2)

= x + y – 3 = 0 and

Y – (-1) = -1(x – 0) = x + y + 1 = 0

Question 11

Find the equation of all lines having slope 2 which are tangents to the curve y = $\frac{1}{x-3}$, x $\neq$ 3.

Solution:-

Equation of the curve y = $\frac{1}{x-3}$ = $(x-3)^{-1}$

$\frac{dy}{dx}$ = (-1)$(x-3)^{-2}$

= $\frac{-1}{(x-3)^2}$

= Slope of the tangent at (x, y)

But according to Question, slope = 2

$\frac{-1}{(x-3)^2} = 2$

$(x-3)^2 = \frac{-1}{2}$

Which is not possible.

Hence, there is no tangent to the given curve having slope 2.

Question 12

Find the equations of all lines having slope 0 which are tangents to the curve y =$\frac{1}{x^2-2x+3}$.

Solution:

Equation of the curve y =$\frac{1}{x^2-2x+3}$. …… (1)

$\frac{dy}{dx} = \frac{d}{dx}[(x^2-2x+3)^{-1}]$

= -$(x^2-2x+3)^{-2}$. (2x – 2)

= $\frac{-2(x-1)}{(x^2-2x+3)^2}$

But according to question, slope = 0, so

= $\frac{-2(x-1)}{(x^2-2x+3)^2} = 0$

= -2(x – 1) = 0

x = 1

From equation (1), y = $\frac{1}{1-2+3} = \frac{1}{2}$

Therefore, the point on the curve which tangent has slope 0 is$\left(1, \frac{1}{2}\right)$.

Equation of the tangent is y - $\frac{1}{2} = 0\ (x-1)$

Y - $\frac{1}{2}$ = 0

Which implies, the value of y is ½.

Question 13

Find the points on the curve $\frac{x^2}{9} + \frac{y^2}{16} = 1$at which the tangents are:

(i) parallel to x – axis **(ii)** parallel to y-axis

Solution:

Equation of the curve $\frac{x^2}{9} + \frac{y^2}{16} = 1$ ………… (1)

Derivate y w.r.t.x, we have

$\frac{2x}{9} + \frac{2y}{16}\frac{dy}{dx} = 0$

$$\frac{2y}{16}\frac{dy}{dx} = -\frac{2x}{9}$$

$\frac{dy}{dx} = \frac{-32x}{18y} = \frac{-16x}{9y}$ ………………. (2)

(i) If tangent is parallel to x – axis, then slope of tangent = 0 which implies, $\frac{dy}{dx}$ = 0

$\frac{-16x}{9y}$ = 0

X = 0

From equation (1), $\frac{y^2}{16}$ = 1

y^2 = 16

Y = $\pm$ 4

The points on curve

(1) Where tangents are parallel to x – axis are (0, $\pm$ 4).

(ii) If the tangent parallel to y – axis.

Slope of the tangent = $\pm\infty$

$\frac{dy}{dx} = \pm\infty$

$\frac{dx}{dy} = 0$ (taking reciprocal)

From equation (2), $\frac{9y}{-16x} = 0$

From equation (1), $\frac{x^2}{9} = 1$

$$x^2 = 9$$

X = $\pm$ 3

Therefore, the points on curve (1) where tangents are parallel to y-axis are $(\pm 3, 0)$

Question 14

Find the equation of the tangents and normal to the given curves at the indicated points:

i. **Y = $x^4 - 6x^3 + 13x^2 - 10x + 5$ at (0, 5)**
ii. **Y = $x^4 - 6x^3 + 13x^2 - 10x + 5$ at (1, 3)**
iii. **Y = x^3 at (1, 1)**
iv. **Y = x^2 at (0, 0)**
v. **X = cos t, y = sin t at t = π/4**

Solution:

(i) Equation of the curve y = $x^4 - 6x^3 + 13x^3 - 10x + 5$

On differentiating y w.r.t. x, we have

$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$

Now, value of $\frac{dy}{dx}\, at\ (0, 5)$

At x = 0,

$4(0)^3 - 18(0)^2 + 26(0) - 10 = -10 = m$ (say)

Slope of the normal at (0, 5) is $\frac{-1}{m} = \frac{-1}{-10} = \frac{1}{10}$

Equation of the tangent at (0, 5) is y – 5 = 10(x – 0)

Y – 5 = 10 x

10x + y = 5

And Equation of the normal at (0, 5) is y – 5 = $\frac{1}{10}(x - 0)$

10 y – 50 = x

x – 10 y + 50 = 0

(ii) Equation of the curve y = $x^4 - 6x^3 + 13x^2 - 10x + 5$

On differentiating y w.r.t. x, we have

$\frac{dy}{dx}$ = $4x^3 - 18x^2 + 26x - 10$

Now value of $\frac{dy}{dx}$ at (1, 3)

At x = 1,

$4(1)^3 - 18(1)^2$ +26(1) – 10 = 4 – 18 + 26 -10 = 2 = m (say)

Slope of the normal at (1, 3) is $\frac{-1}{m} = \frac{-1}{2}$

Equation of the tangent at (1, 3) is y – 3 = 2 (x – 1)

Y – 3 = 2x – 2

Y = 2x + 1

Slope of the normal at (1, 3) is $\frac{-1}{m} = \frac{-1}{2}$
Equation of the tangent at (1, 3) is y – 3 = 2(x – 1)
Y – 3 = 2x -2
Y = 2x + 1
And Equation of the normal at (1, 3) is y -3 = $\frac{-1}{2}$ (x – 1)
2y – 6 = -x + 1
x + 2y – 7 = 0

(iii) Equation of the curve y = x^3 ……. (1)
On differentiating y w.r.t. x, we have

$\frac{dy}{dx} = 3x^2$

Now, value of $\frac{dy}{dx}$ at (1, 1)

At x = 1,

$3(1)^2 = 3 = m$ (say)

Slope of the normal at (1, 1) is $\frac{-1}{m} = \frac{-1}{3}$
Equation of the tangent at (1, 1) is y – 1 = 3 (x – 1)
Y – 1 = 3x – 3 or y = 3x – 2
And Equation of the normal at (1, 1) is y – 1 = $\frac{-1}{3}(x - 1)$
3y – 3 = -x +1
x + 3y – 4 = 0

(iv) Equation of the curve y = x^2 .... (1)
On differentiating y w.r.t. x, we have

$$\frac{dy}{dx} = 2x$$

Now value of $\frac{dy}{dx}$ at (0, 0)
At x = 0, 2 × 0 = 0 = m (say)
Equation of the tangent at (0, 0) is y – 0= 0 (x- 0)
Y = 0
And normal at (0, 0) is y – axis.

(v) Equation of the curves are x = cos t, y = sin t
$\therefore \frac{dx}{dt}$ = - sin t and $\frac{dy}{dt}$ = cos t
$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{-\sin t} = -\cot t$
Slope of the tangent at t = $\frac{\pi}{4}$ = -cot $\frac{\pi}{4}$ = -1 = m (say)
Slope of the normal at t = $\frac{\pi}{4}$ is $\frac{-1}{m} = \frac{-1}{-1} = 1$
Point (x, y) = (cos t, sin t)
= $\left(\cos\frac{\pi}{4}, \sin\frac{\pi}{4}\right)$
= $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
Equation of the tangent is y - $\frac{1}{\sqrt{2}}$ = -1$\left(x - \frac{1}{\sqrt{2}}\right)$
x + y = $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$
x + y = $\sqrt{2}$
And equation of the normal is y - $\frac{1}{\sqrt{2}}$ = 1 $\left(x - \frac{1}{\sqrt{2}}\right)$
y - $\frac{1}{\sqrt{2}} = x - \frac{1}{\sqrt{2}}$

⇨ y = x

Question 15

Find the equation of the tangent line to curve y = $x^2 - 2x + 7$ which is:

(a) Parallel to the line 2x – y + 9 = 0

(b) Perpendicular to the line 5y – 15x = 13

Solution:-

Equation of the curve y = $x^2 - 2x + 7$ …….. (1)

Slope of the tangent = $\frac{dy}{dx}$ = 2x – 2 …………… (2)

(a) Slope of the line 2x – y + 9 = 0 is $\frac{-a}{b} = \frac{-2}{-1}$ = 2

Slope of tangent parallel to this line is also = 2

From equation (2), 2x – 2 = 2

⇨ X = 2

From equation (1), y = 4 – 4 + 7 = 7

Therefore, point of contact is (2, 7).

Equation of the tangent at (2, 7) is y – 7 = 2 (x – 2)

⇨ Y – 7 = 2x – 4

⇨ Y – 2x – 3 = 0

(b) Slope of the line -15x + 5y = 13 is $\frac{-a}{b}$ = $\frac{-(-15)}{3}$ = 3

Slope of the required tangent perpendicular to this line = $\frac{-1}{m} = \frac{-1}{3}$

From equation (2), 2x – 2 = $\frac{-1}{3}$

6x – 6 = -1

X = $\frac{5}{6}$

From equation (1), y = $\frac{25}{36} - \frac{5}{3} + 7$

= $\frac{25-60+252}{36}$ = $\frac{217}{36}$

Therefore, point of contact is $\left(\frac{5}{6}, \frac{217}{36}\right)$

Equation of the required tangent is y = $\frac{217}{36} = \frac{-1}{3}\left(x - \frac{5}{6}\right)$

3y - $\frac{217}{12}$ = -x + $\frac{5}{6}$

X + 3y = $\frac{217}{12}$ + $\frac{5}{6}$

X + 3y = $\frac{217+10}{12}$ = $\frac{227}{12}$

12x + 36 y = 227 (which is required equation)

Question 16

Show that the tangents to the curve y = $7x^3 + 11$ at the points where x = 2 and x = -2 are parallel.

Solution:-

Equation of the curve y = $7x^3 + 11$

Slope of tangent at (x, y) = $\frac{dy}{dx}$ = $21x^2$

At the point x = -2

Slope of the tangent = $21(-2)^2$ = 21 × 4 = 84

Since, the slopes of the two tangents are equal.

Therefore, tangents at x = 2 and x = -2 are parallel.

Question 17

Find the points on the curve y = x^3at which the slope of the tangent is equal to the y- coordinate of the point.

Solution:-

Equation of the curve y = x^3 ……….. (1)

Slope of tangent at (x, y)

= $\frac{dy}{dx}$ = $3x^2$ …… (2)

As given, Slope of the tangent = y-coordinate of the point

$3x^2 = x^3$

$3x^2 - x^3 = 0$

x^2 (3 – x) = 0

x^2 =0 or 3 – x = 0

x = 0 or x = 3

From equation (1), at x= 0, y = 0

The point is (0, 0)

And from equation (1), at x = 3, y = 27

The point is (3, 27)

Therefore, the desired points are (0, 0) and (3, 27)

Question 18

For the curve y =$4x^3 - 2x^3$, find all point at which the tangent passes through the origin.

Solution:-

Equation of the curve y = $4x^3 - 2x^3$ ………. (1)

Slope of the tangent at (x, y) passing through origin (0, 0)

$\frac{dy}{dx} = 12x^2 - 10x^4$

And dy/dx = $\frac{y-0}{x-0}$

⇨ $\frac{y}{x} = 12x^2 - 10x^4$

$Y = 12x^3 - 10x^5$

Substituting value of y in equation (1), we get.

$12x^3 - 10x^5 = 4x^3 - 2x^5$

$8x^3 - 8x^5 = 0$

$8x^3(1 - x^2) = 0$

$8x^2 = 0$ or $1 - x^2 = 0$

⇨ $x = 0$ or $x = \pm 1$

From equation (1),

at x = 0, the value of y = 0

From equation (1), at x = 1,

The value of y is, y = 4 - 2 = 2

From equation (1), at x = -1,

Therefore, the required points are (0, 0), (1, 2) and (-1, -2).

Question 19

Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to x-axis.

Solution:-

Equation of the curve $x^2 + y^2 - 2x - 3 = 0$ ……….. (1)

On differentiating expression w.r.t.x, we have

$2x + 2y\frac{dy}{dx} - 2 = 0$

$2y\frac{dy}{dx} = 2 - 2x$

$\frac{dy}{dx} = \frac{2(1-x)}{2y} = \frac{1-x}{y}$

Since tangent is parallel to x- axis: $\frac{dy}{dx} = 0$

$\frac{1-x}{y} = 0$

⇨ $x = 1$

From equation (1), $1 + y^2 - 2 - 3 = 0$

⇨ $y^2 = 4$

⇨ $y = \pm 2$

Therefore, the required points are (1, 2) and (1, -2)

Question 20

Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$

Solution:-

Equation of the curve $ay^2 = x^3$ .... (1)

On differentiating expression w.r.t.x, we have

$a\frac{d}{dx}y^2 = \frac{d}{dx}x^3$

$a.2y\frac{dy}{dx} = 3x^2$

$$\frac{dy}{dx} = \frac{3x^2}{2ay}$$

Slope of the tangent at the point $(am^2, am^3) = \frac{-2}{3m}$

Equation of the normal at (am^2, am^3) is

$y - am^3 = \frac{-2}{3m}(x - am^2)$

⇨ $3my - 3am^4 = -2x + 2am^2$

⇨ $2x + 3my - 2am^2 - 3am^4 = 0$

⇨ $2x + 3my - am^2(2 + 3m^2) = 0$

Question 21

Find the equations of the normal to the curve y = $x^3 + 2x + 6$ which are parallel to the line x + 14 y + 4 = 0.

Solution:-

Equation of the curve y = $x^3 + 2x + 6$ ……. (1)

Slope of the tangent at (x, y)

So, $\frac{dy}{dx}$ = $3x^2 + 2$

Slope of the normal to the curve at (x, y)

= $\frac{-1}{3x^2+2}$ ………. (2)

Since Slope of the normal = $\frac{-1}{14}$ (Given)

$\frac{1}{3x^2+2}$ = $\frac{-1}{14}$

⇨ $3x^2 + 2 = 14$

⇨ $3x^2$ +2 = 14

$3x^2$ = 12

x^2 = 4

$$x = \pm 2$$

From equation (1), at x = 2, y = 8 + 4 + 6 = 18

At x = -2, y = -8-4+6 = -6

Therefore, the points of contact are (2, 16) and (-2, -6).

Equation of the normal at (2, 18) is y – 18 = $\frac{-1}{14}$(x - 2)

⇨ 14y – 252 = -x + 2

x + 14 y - 254 = 0

And equation of the normal at (-2, -6) is y + 6 = $\frac{-1}{14}$ (x + 2)

14 y + 84 = -x -2

X + 14 y – 86 = 0

And Equation of the normal at (-2, -6) is y + 6 = $\frac{-1}{14}$(x + 2)

14y + 84 = -x – 2

x + 14y + 86 = 0

Which is required equation.

Question 22

Find the equation of the tangent and normal to the parabola y^2 = 4ax at the point$(at^2 . 2at)$.

Solution-

Equation of the parabola $y^2 = 4ax$ ………… (1)

Slope of the tangent at (x, y)

$= \frac{dy}{dx} y^2 = 4a \frac{d}{dx}(x)$

⇨ $2y\frac{dy}{dx} = 4a$

⇨ $\frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}$

Slope of the tangent at the point $(at^2, 2at) = \frac{2a}{2at} = \frac{1}{t}$

Slope of the normal = -t

Equation of the tangent at the point $(at^2, 2at)$

$= y - 2at = \frac{1}{t}(x - at^2)$

$ty - 2at^2 = x - at^2$

$ty = x + at^2$

And Equation of the normal at the point $(at^2, 2at)$

$= y - 2at = -t\,(x - at^2)$

Which implies, $tx + y = 2at + at^3$

Question 23

Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$.

Solution:-

Equations of the curves are $x = y^2$ …….. (1) and

$xy = k$ ………… (2)

Substituting the value of x in equation (2), we get $y^2 . y = k$

$Y = k^{1/3}$

Put the value of x in equation (1), we get

$X = \left(k^{1/3}\right)^2 = k^{2/3}$

Therefore, the point of intersection (x, y) is

$\left(k^{2/3}, k^{1/3}\right)$ ………… (3)

Differentiating equation (1) w.r.t.x

⇨ $\frac{dy}{dx} = \frac{1}{2y} = m_1$ ……. (4)

Differentiating equation (2) w.r.t

⇨ $\frac{dy}{dx} = \frac{-y}{x} = m_2$ ……. (5)

According to the question, $m_1 m_2 = -1$

Which implies,

$\frac{1}{2y}\left(\frac{-y}{x}\right) = -1$

$\frac{1}{2x} = 1$

$2x = 1$

$2k^{1/3} = 1$ [using equation (3)]

Taking cube both the sides,

$$8k^2 = 1$$

Hence proved.

Question 24

Find the equation of the tangent and the normal to the hyperbola $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point(x_0, y_0).

Solution:

Equation of the hyperbola $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.............. (1)

On differentiating w.r.t. x, we get

$$\frac{2x}{a^2} - \frac{2y}{b^2}.\frac{dy}{dx} = 0$$

$\frac{-2y}{b^2}.\frac{dy}{dx} = \frac{-2x}{a^2}$

$\frac{dy}{dx} = \frac{b^2x}{a^2y}$ .... (2)

Slope of tangent at (x_0, y_0) is y - y_0 = $\frac{b^2x_0}{a^2y_0}$

Equation of the tangent at (x - x_0) is$y - y_0 = \frac{b^2x_0}{a^2y_0}(x - x_0)$

$yy_0 - {y_0}^2 = \frac{b^2}{a^2}(xx_0 - {x_0}^2)$

$\frac{yy_0}{b^2} - \frac{{y_0}^2}{b^2} = \frac{xx_0}{a^2} - \frac{{x_0}^2}{a^2}$

$\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = \frac{{x_0}^2}{a^2} - \frac{{y_0}^2}{b^2}$ .... (3)

Since (x_0, y_0) lies on the hyperbola (1), therefore, $\frac{{x_0}^2}{a^2} - \frac{{y_0}^2}{b^2} = 1$

From equation (3), $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$

Now, slope of normal at (x_0, y_0) = $\frac{-a^2y_0}{b^2x_0}$

Therefore,

Equation of the normal at (x_0, y_0) = $\frac{-a^2x_0}{b^2y_0}(x - x_0)$

b^2x_0y - $b^2x_0y_0 = a^2y_0x + a^2x_0y_0$

$$b^2x_0(y - y_0) = -a^2y_0(x - x_0)$$

On dividing both sides by $a^2b^2x_0y_0$, we get

$\frac{y-y_0}{a^2y_0}$ = - $\frac{(x-x_0)}{b^2x_0}$

$\frac{(x-x_0)}{b^2x_0} + \frac{y-y_0}{a^2y_0} = 0$

Which is required equation.

Question 25

Find the equation of the tangent to the curve y = $\sqrt{3x-2}$ which is parallel to the line 4x – 2y + 5 = 0.

Solution:-

Equation of the curve y = $\sqrt{3x-2}$ ……….. (1)

Slope of the tangent at point (x, y) is $\frac{dy}{dx} = \frac{d}{dx}(3x-2)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(3x-2)^{\frac{1}{2}}\frac{d}{dx}(3x-2)$

= $\frac{1}{2\sqrt{3x-2}}$. 3 …………… (2)

Again slope of the 4x – 2y + 5 = 0 is $\frac{-a}{b} = \frac{-4}{-2}$ = 2 …………… (3)

As given: parallel lines have same slope

By equation slopes of both the lines, we get

= $\frac{1}{2\sqrt{3x-2}}$. 3 = 2

4 $\sqrt{3x-2}$ = 3

16(3x – 2) = 9

48x – 32 = 9

48x = 41

x = $\frac{41}{48}$

Substitute the value of x in equation (1).

Y = $\sqrt{3\left(\frac{41}{48}\right) - 2}$

= $\sqrt{\frac{41}{16} - 2}$

= $\sqrt{\frac{41-32}{16}}$ = $\sqrt{\frac{9}{16}}$ = $\frac{3}{4}$

Therefore, point of contact is$\left(\frac{41}{48}, \frac{3}{4}\right)$.

Now,

Equation of the required tangent is y - $\frac{3}{4} = 2\left(x - \frac{41}{48}\right)$

y - $\frac{3}{4} = 2x - \frac{41}{24}$

y = 2x + $\frac{18-41}{24}$

24y = 48x – 23

48x – 24 y = 23

Which is required equation.

Choose the correct answer in Exercise 26 and 27

Question 26

The slope of the normal to the curve y = 2x^2 + 3 sin x at x = 0 is:

a) 3

b) 1/3

c) -3

d) -1/3

Solution:-

Option (d) is correct

Explanation:

Equation to the curve y = $2x^2 + 3 \sin x$ ………. (1)

Slope of the tangent at point (x, y) is $\frac{dy}{dx} = 4x + 3 \cos x$

Slope of the tangent at x = 0, 4(0) + 3 cos 0 = 3 = m (say)

Slope of the normal = $\frac{-1}{m} = \frac{-1}{3}$

Question27

The line y x+1 is a tangent to the curve $y^2 = 4x$ at the point:
(1, 2) (b) (2, 1) (c) (1,-2) (d) (-1, 2)

Solution:
Option (A) is correct.

Explanation:
Slope of the tangent at point (x, y) is $2y\frac{dy}{dx} = 4$
$\frac{dy}{dx} = \frac{2}{y}$ ……………. (2)
[as we know $\frac{-a}{b} = \frac{-1}{-1} = 1$]
From equation (2) and (3)

$$\frac{2}{y} = 1$$
$$y = 2$$

From equation (1), 4 = 4x
X = 1
Therefore, required point is (1, 2)

Exercise6.4

Question1

Using differentials, find the approximate value of each of the following up to 3 places of decimal:

i. $\sqrt{25.3}$
ii. $\sqrt{49.5}$
iii. $\sqrt{0.6}$
iv. $(0.009)^{1/3}$
v. $(0.999)^{1/10}$
vi. $(15)^{1/4}$
vii. $(26)^{1/3}$
viii. $(255)^{1/4}$
ix. $(82)^{1/4}$
x. $(401)^{1/2}$
xi. $(0.0037)^{1/2}$
xii. $(26.57)^{1/3}$
xiii. $(81.5)^{1/4}$
xiv. $(3.968)^{3/2}$
xv. $(32.15)^{1/5}$

Solution:

(i) $\sqrt{\mathbf{25.3}}$

Consider, $y = \sqrt{x}$ ……………. (1) and then

$y+\Delta y = \sqrt{x + \Delta x}$

On differentiating equation (1) w.r.t.x, we get

$\frac{dy}{dx} = \frac{1}{2} x^{\frac{-1}{2}} = \frac{1}{2\sqrt{x}}$

$\Rightarrow dy = \frac{dx}{2\sqrt{x}}$ ………….. (2)

Now, given expression can be written as,

$\sqrt{25.3} = \sqrt{25 + 0.3}$

Here, x=25 and$\Delta x = 0.3$, then $\Delta y = \sqrt{x + \Delta x} - \sqrt{x}$

$= \sqrt{25.3} - \sqrt{25} = \sqrt{25.3} - 5$

$\Rightarrow \sqrt{25.3} = \Delta y + 5$

Since, Δx and Δy is approximately equal to dx anddy respectively.

From equation (2),

$dy = \frac{0.3}{2\sqrt{25}}$

= 0.03

Hence, approximately value of $\sqrt{25.3}$ is 5+0.03 =5.03.

(ii)$\sqrt{49.5}$

Consider, y=$\sqrt{x}$ …………… (1)

On different equation (1) w.r.t. x, we get

$\frac{dy}{dx} = \frac{1}{2}x^{\frac{-1}{2}} = \frac{1}{2\sqrt{x}}$

$\Rightarrow dy = \frac{dx}{2\sqrt{x}}$ ………… (2)

Now, from equation (1),$y + \Delta y = \sqrt{x + \Delta x}$

= $\sqrt{49.5}$ = $\sqrt{49 + 0.5}$

Here, x=49 and $\Delta x = 0.5$, then

$\Delta y = \sqrt{x + \Delta x}$ -$\sqrt{x}$

= $\sqrt{49.5} - \sqrt{49} = \sqrt{49.7} - 7$

$\Rightarrow\sqrt{49.5} = \Delta y + 7$

Since, Δx and Δy is approximately equal to dx and dy respectively.

From equation (2), dy =$\frac{0.5}{2\sqrt{49}}$

= 0.0357

So, approximately value of $\sqrt{49.5}$ is 7+0.00357= 7.0357

(iii)$\sqrt{0.6}$

Consider, y = $\sqrt{x}$………… (1)

On differentiating equation (1) w.r.t. x, we get

$\frac{dx}{dy} = \frac{1}{2}x^{\frac{-1}{2}} = \frac{1}{2\sqrt{x}}$

$dy = \frac{dx}{2\sqrt{x}}$ ………… .(2)

Now, from equation (1), $y + \Delta y = \sqrt{x + \Delta x}$

= $\sqrt{0.6} - \sqrt{0.64 - 00.4}$

Here, $x = 0.64$ and$\Delta x = -0.04$, then

$\Delta y = \sqrt{x + \Delta x} - \sqrt{x}$

= $\sqrt{0.6} - \sqrt{0.64} = \sqrt{0.6} - 0.8$

$\sqrt{0.6} = \Delta y + 0.8$

Since, Δx and Δy is approximately equal todxand dy respectively.

From equation (2), $dy = \frac{-0.04}{2\sqrt{0.64}} = -0.025$

Therefore, approximately value of $\sqrt{0.6}$ is 0.8-0.025=0.775.

(iv)$(0.009)^{\frac{1}{3}}$

Consider, $y = \sqrt[3]{x}$………….. (1)

On differentiating equation (1) w.r.t. x, we get

$\frac{dy}{dx} = \frac{1}{3}x^{\frac{-2}{3}} = \frac{1}{3x^{2/3}}$

$dy = \frac{dx}{3x^{2/3}} = \frac{dx}{3\left(x^{1/3}\right)^2}$ ………… (2)

Now, from equation (1), $y = \Delta y = \sqrt{x + \Delta x}$

$=(0.009)^{\frac{1}{3}} = (0.008 + 0.001)^{\frac{1}{3}}$

Here, $x = 0.008$ and$\Delta x = 0.001$,

Then$\Delta y = \sqrt{x + \Delta x} - \sqrt{x}$

$= (0.009)^{\frac{1}{3}} - (0.008)^{\frac{1}{3}} = (0.009)^{\frac{1}{3}} - 0.2$

$(0.009)^{\frac{1}{3}} = \Delta y + 0.2$

Since, Δx and Δy is approximately equal to dx and dy respectively.

From equation

$dy = \frac{0.001}{3\left((0.008)^{\frac{1}{3}}\right)^2}$

(2), $\frac{0.001}{3\times 0.04} = 0.0083$

There, approximately value of $(0.009)^{\frac{1}{3}}$ is 0.2+0.0083= 0.2.83

(v) $(0.999)^{\frac{1}{10}}$

Consider, $y = x^{\frac{1}{10}}$……………. (1)

On differentiating equation (1) w.r.t. x, we get

$\frac{dy}{dx} = \frac{1}{10}x^{\frac{-9}{10}} = \frac{1}{10x^{\frac{9}{10}}}$

$dy = \frac{dx}{10\left(x^{\frac{1}{10}}\right)^9}$ …………….. (2)

Now , from equation (1), $y + \Delta y = \sqrt{x + \Delta x}$

$= (0.999)^{\frac{1}{10}} = (1 - 0.001)^{\frac{1}{10}}$………………. (3)

Here $x = 1$ and $\Delta x = -0.001$

Then $\Delta y = (x + \Delta x)^{\frac{1}{10}}$-$x^{\frac{1}{10}}$

$= (0.999)^{\frac{1}{10}} - 1$

$= (0.999)^{\frac{1}{10}} = 1 + \Delta y$

Since, Δx and Δy is approximately equal to dx and dy respectively.

From equation (2),

$dy = \frac{-0.001}{10\left(1^{\frac{1}{10}}\right)^9}$ =-00001

Therefore, approximate value of $(0.999)^{\frac{1}{10}}$ is 1 -0.0001=0.9999.

(vi) $(15)^{\frac{1}{4}}$

Consider, $y = x^{\frac{1}{4}}$………….. (1)

On differentiating equation (1) w.r.t. x, we get

$\frac{dy}{dx} = \frac{1}{4}x^{\frac{-3}{4}} = \frac{1}{4x^{\frac{3}{4}}}$

$\Rightarrow dy = \frac{dx}{4\left(x^{\frac{1}{4}}\right)^3}$ ……………… (2)

Now, from equation (1), $y + \Delta y = \sqrt{x + \Delta x}$

$= (15)^{\frac{1}{4}} = (16 - 1)^{\frac{1}{4}}$…………… (3)

Here $x = 16$ and $\Delta x = -1$

Then $\Delta y = (x + \Delta x)^{\frac{1}{4}} - x^{\frac{1}{4}}$

$= (15)^{\frac{1}{4}} - 16^{\frac{1}{4}} = (15)^{\frac{1}{4}} - 2$

$\Rightarrow (15)^{\frac{1}{4}} = 2 + \Delta y$

Since, Δx and Δy is approximately equal to dx and dy respectively.

$dy = \frac{-1}{4\left(16^{\frac{1}{4}}\right)^3} = \frac{-1}{32}$

From Equation (2),

Therefore, approximate value of $(15)^{\frac{1}{4}}$ is $2 - \frac{1}{32} = \frac{63}{32}$=1.96875.

(vii) $(26)^{\frac{1}{3}}$

Consider, $y = \sqrt[3]{x}$……………. (1)

On differentiating equation (1) w.r.t. x, we get

$\frac{dy}{dx} = \frac{1}{3}x^{\frac{-2}{3}} = \frac{1}{3x^{2/3}}$

$dy = \frac{dx}{3x^{2/3}} = \frac{dx}{\left(3x^{1/3}\right)^2}$ ……………… (2)

Now, from equation (1), $y + \Delta y = \sqrt{x + \Delta x}$

$= 26^{\frac{1}{3}} = (27 - 1)^{\frac{1}{3}}$

Here, $x = 27$ and $\Delta x = -1$

Then $\Delta y = \sqrt{x + \Delta x} - \sqrt{x} = (26)^{\frac{1}{3}} - (27)^{\frac{1}{3}} = (26)^{\frac{1}{3}} - 3$

$(26)^{\frac{1}{3}} = \Delta y + 3$

Since, Δx and Δy is approximately equal todx and dy respectively.

From equation (2)

$dy = \frac{-1}{3\left((27)^{\frac{1}{3}}\right)^2}$

$= \frac{-1}{27}$

Therefore, approximately value of $(26)^{\frac{1}{3}}$ is $3 - \frac{1}{27} = \frac{80}{27}$= 2.9629.

(viii) $(255)^{\frac{1}{4}}$

Consider, $y = x^{\frac{1}{4}}$................. (1)

On differentiating equation (1) w.r.t. x, we get

$\frac{dy}{dx} = \frac{1}{4}x^{\frac{-3}{4}} = \frac{1}{4x^{\frac{3}{4}}}$

$dy = \frac{dx}{4\left(x^{\frac{1}{4}}\right)^3}$ .... (2)

Now, from equation (1), $y + \Delta y = \sqrt{x + \Delta x}$

$=(15)^{\frac{1}{4}} = (256 - 1)^{\frac{1}{4}}$.................. (3)

Here $x = 256$ and $\Delta x = -1$

Then $\Delta y = (x + \Delta x)^{\frac{1}{4}} - x^{\frac{1}{4}}$

$= (255)^{\frac{1}{4}} - (256)^{\frac{1}{4}} = (255)^{\frac{1}{4}} - 4$

$(255)^{\frac{1}{4}} = 4 + \Delta y$

Since, Δx and Δy is approximately equal to dx and dy respectively.

$dy = \frac{-1}{4\left(256^{\frac{1}{4}}\right)^3} = \frac{-1}{256}$

From Equation (2),

Therefore, approximate value of $(255)^{\frac{1}{4}}$ is $4 - \frac{1}{256} = \frac{1023}{256}$ =3.9961

(ix) $(82)^{\frac{1}{4}}$

Consider, $y = x^{\frac{1}{4}}$ .... (1)

On differentiating equation (1) w.r.t. x, we get

$\frac{dy}{dx} = \frac{1}{4}x^{\frac{-3}{4}}$

$= \frac{1}{4x^{\frac{3}{4}}}$

$dy = \frac{dx}{4\left(x^{\frac{1}{4}}\right)^3}$ .... (2)

Now, form equation (1), $y + \Delta y = \sqrt{x + \Delta x} = (82)^{\frac{1}{4}} = (81 + 1)^{\frac{1}{4}}$ .... (3)

Here $x = 81$ and $\Delta x = 1$

Then $\Delta y = (x + \Delta x)^{\frac{1}{4}} - x^{\frac{1}{4}}$

$= (82)^{\frac{1}{4}} - (81)^{\frac{1}{4}} = (82)^{\frac{1}{4}} - 3$

$(82)^{\frac{1}{4}} = 3 + \Delta y$

Since, Δx and Δy is approximately equal to dx and dy respectively.

$dy = \frac{1}{4\left(81^{\frac{1}{4}}\right)^3} = \frac{1}{108}$

From equation (2),

Therefore, approximately value of $(82)^{\frac{1}{4}}$ is $3 + \frac{1}{108} = \frac{325}{108}$ = 3.0092

(x) $\sqrt{401}$

Consider, $y = \sqrt{x}$ ……………… (1)

On differentiating equation (1) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2}x^{\frac{-1}{2}}\frac{1}{2\sqrt{x}}$$

$dy = \frac{dx}{2\sqrt{x}}$ ……………… (2)

Now, from equation (1), $y + \Delta y = \sqrt{x + \Delta x}$

$= \sqrt{401} = \sqrt{400 + 1}$

Here, $x = 400$ and $\Delta x = 1$, then $\Delta y = \sqrt{x + \Delta x}$

$= \sqrt{401} - \sqrt{400} = \sqrt{401} - 20$

$\Rightarrow \sqrt{401} = \Delta y + 20$

Since, Δx and Δy is approximately equal to dx and dy respectively.

From equation (2), $dy = \frac{1}{2\sqrt{400}} = \frac{1}{40}$

Therefore, approximately value of $\sqrt{401}$ is $20 + \frac{1}{40} = \frac{801}{40} = 20.025$

(xi) $\sqrt{0.0037}$

Consider, $y = \sqrt{x}$ ………………… (1)

On differentiating equation (1), w. r. t. x, we get

$$\frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$dy = \frac{dx}{2\sqrt{x}}$ ………………….. (2)

Now, from equation (1), $y + \Delta y = \sqrt{x + \Delta x}$

$= \sqrt{0.0037} = \sqrt{0.0036 + 0.0001}$

Here, $x = 0.0036$ and $\Delta x = 0.0001$, then

$\Delta y = \sqrt{x + \Delta x} - \sqrt{x}$

$= \sqrt{0.0037} - \sqrt{0.0036} = \sqrt{0.0037} - 0.06$

$\sqrt{0.0037} = \Delta y + 0.006$

Since, Δx and Δy is approximately equal to dx and dy respectively.

$$dy = \frac{0.0001}{2\sqrt{0.0036}}$$

From equation (2),

$$= \frac{0.0001}{0.12}$$

Therefore, approximately value of $\sqrt{0.0037}$ is $0.06 + \frac{0.0001}{0.12} = 0.060833$

(xii) $(26.57)^{\frac{1}{3}}$

Consider, $y = \sqrt[3]{x}$………………….. (1)

On differentiating equation (1) w.r.t. x, we get

$\frac{dy}{dx} = \frac{1}{3}x^{\frac{-2}{3}} = \frac{1}{3x^{2/3}}$

$dy = \frac{1}{3x^{2/3}} = \frac{1}{\left(3x^{2/3}\right)^2}$ ……………… (2)

Now, from equation (1), $y + \Delta y = \sqrt{x + \Delta x}$

$= (26.57)^{\frac{1}{3}} = (27 - 0.43)^{\frac{1}{3}}$

Here, $x = 27$ and $\Delta x = -0.43$,

Then $\Delta y = \sqrt{x + \Delta x} - \sqrt{x}$

$= (26.57)^{\frac{1}{3}} - (27)^{\frac{1}{3}} = (26.57)^{\frac{1}{3}} - 3$

$(26.57)^{\frac{1}{3}} = \Delta y + 3$

Since, Δx and Δy is approximately equal to dx and dy respectively.

$dy = \frac{-0.43}{3\left((27)^{\frac{1}{3}}\right)^2}$

From equation (2),

$=\frac{-0.43}{27} = 0.0159$

Therefore, approximately value of $(26.57)^{\frac{1}{3}}$ is 3-0.001590 = 2.9841.

(xiii) $(81.5)^{\frac{1}{3}}$

Consider, $y = x^{\frac{1}{4}}$ …………….. (1)

On differentiating equation (1) w.r.t. x, we get

$\frac{dy}{dx} = \frac{1}{4}x^{\frac{-3}{4}} = \frac{1}{4x^{\frac{3}{4}}}$

$dy = \frac{dx}{\left(x^{\frac{1}{4}}\right)^3}$ ……………….. (2)

Now, from equation (1), $y + \Delta y = \sqrt{x + \Delta x}$

$= (81.5)^{\frac{1}{4}} = (81 + 0.5)^{\frac{1}{4}}$ …………… (3)

Here $x = 81$ and $\Delta x = 0.5$

Then $\Delta y = (x + \Delta x)^{\frac{1}{4}} - x^{\frac{1}{4}}$

$= (81.5)^{\frac{1}{4}} - (81)^{\frac{1}{4}} = (81.5)^{\frac{1}{4}} - 3$

- $(81.5)^{\frac{1}{4}} = 3 + \Delta y$

Since, Δx and Δy is approximately equal to dx and dy respectively.

$dy = \frac{0.5}{4\left(81^{\frac{1}{4}}\right)^3} = \frac{0.5}{108}$

From equation (2),

= 0.00462

Therefore, approximately value of $(82)^{\frac{1}{4}}$ is 3 + 0.00462

(xiv) $(3.968)^{\frac{3}{2}}$

Consider, $y = x^{\frac{3}{2}} = x^{1+\frac{1}{2}} = x\sqrt{x}$ .... (1)

$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}$

$dy = \frac{3\sqrt{x}}{2}\ dx$ .... (2)

Now, from equation (1), $y + \Delta y = \sqrt{x + \Delta x}$

$= (3.968)^{\frac{3}{2}} = (4 - 0.032)^{\frac{3}{2}}$

Here, $x = 4$ and $\Delta x = 0.032$, then $\Delta y = \sqrt{x + \Delta x} - \sqrt{x}$

$(3.968)^{\frac{3}{2}} - (4)^{\frac{3}{2}} = (3.968)^{\frac{3}{2}} - 8$

$(3.968)^{\frac{3}{2}} = \Delta y + 8$

Since, Δx and Δy is approximately equal to dx and dy respectively.

From equation (2),

$dy = \frac{3}{2}\sqrt{4}(-0.032) = -0.096$

Therefore, approximately value of $(3.968)^{\frac{3}{2}}$ is $8 - 0.096 = 7.904$

(xv) $(32.15)^{\frac{1}{5}}$

Consider, $y = x^{\frac{1}{3}}$................ (1)

On differentiating equation (1) w.r.t. x, we get

$\frac{dy}{dx} = \frac{1}{5}x^{\frac{-4}{5}} = \frac{1}{5x^{\frac{4}{5}}}$

$dy = \frac{dx}{\left(x^{\frac{1}{5}}\right)^4}$ .... (2)

Now from equation (1), $y + \Delta y = \sqrt{x + \Delta x}$

$= (32.15)^{\frac{1}{5}} = (32 + 0.15)^{\frac{1}{5}}$............ (3)

Here $x = 32$ and $\Delta x = 0.15$

Then$\Delta y = (x + \Delta x)^{\frac{1}{5}} - x^{\frac{1}{5}}$

$$= (32.15)^{\frac{1}{5}} - (32)^{\frac{1}{5}} = (32.15)^{\frac{1}{5}} - 2$$

$$(32.15)^{\frac{1}{5}} = 2 + \Delta y$$

Since, Δx and Δx is approximately equal to dx and dy respectiley.

From equation (2),

$dy = \frac{0.15}{5\left(32^{\frac{1}{5}}\right)^4} = \frac{0.15}{80} = 0.001875$

Therefore, approximately value of $(32.15)^{\frac{1}{5}}$ is 2+0.001875 =2.001875

Question2

Find the approximately value of $f(2.01)$ where $f(x) = 4x^2 + 5x + 2$,

Solution:

Consider, $f(x) = y = x^3 + 5x + 2$ .... (1)
On differentiating equation (1) w.r.t. x, we get
$f(x) = \frac{dy}{dx} = 8x + 5$
$dy = (8x + 5)dx$
$= (8x+5)\,\Delta x$ .... (2)
Changing x to $x + \Delta x$ and y to $y + \Delta y$ in equation (1),
$y + \Delta y = f(x + \Delta x)$
$= f(2.01) = f(2 + 0.01)$ .... (3)
Here, $x = 2$ and $\Delta x = 0.01$
From equation (3), $f(2.01) = y + \Delta y$
Since, Δx and $\Delta y$is approximately equal to dx and dy respectively.
From equation (1) and (2), we get
$f(2.01) = (4x^2 + 5x + 2) + (8x + 5)\Delta x$
Therefore
$f(2.01) = 4(4) + 5(2) + 2 + (8 \times 2 + 5)(0.01) = 28.21$
Therefore, approximately value of f(2.01) is 28.21.

Quetion3

Find the approximately value of $f(5.001)$ where $f(x) = x^3 - 7x^3 + 15$.

Solution:

Consider, $f(x) = y = x^3 - 7x^2 + 15$.................. (1)
On differentiating equation (1) w.r.t. x, we get
$f'(x) = \frac{dy}{dx} = 3x^2 - 14x$
Or $dy = (3x^2 - 14x)dx(3x^2 - 14x)\Delta x$ .... (2)
Changing xtox $+ \Delta x$ and y to $y + \Delta y$ n equation (1),
$y + \Delta y = f(x + \Delta x)$
$= f(5.001) = f(5 + 0.001)$ .... (3)
Here, $x = 5$ and $\Delta x = 0.001$
From equation (3), $f(5.001) = y + \Delta y$
Since, Δx and Δy is approximately equal to dx and dy respestively.
From equation (1) and (2),
$f(5.001) = (x^{3-}7x^2 + 15) + (3x^2 - 14x)\Delta x$
$f(5.001) = 125 - 175 + 15 + (75 - 70)(0.001)$
$= -35 + 0.005 = -34.995$
Therefore, approximately value of f(5.001) is -34.995

Question4

Find the approximate change in the volume of a cube of side x meters caused by increasing the side by 1%.

Solution:

Side of a cube is x meters then cube volume (V) is $x^{3.}$
Or $V=x^3$ ………… (1)
On differentiating equation (1) w.r.t. x, we get
$\frac{dV}{dx} = 3x^2$ ……… (2)
According to the
Statement, increase in side =1% $=\frac{x}{100}$
So, $\Delta x \frac{x}{100}$ ………… (3)
Approximately change in volume, V, of cube= $\Delta V \sim dV$
$=\frac{dV}{dx} dx$
$=\frac{dV}{dx} \Delta x$
$\sim 3x^2 \left(\frac{x}{100}\right) \sim \frac{3}{100} x^3 \sim 0.03x^3$ Cubic meters

Question5

Find the approximately change in the surface area of a cube of side x meters caused by decreasing the side by 1%.

Solution:

Side of a cube is x meters then Surface area of a cube is (s) =6x2
S=6x2
On differentiating above equation w.r.t. x, we get
$\frac{dS}{dx} = 12x$
At per question, if decrease in side 1%is
=-1% of x
= -0.001x
$\Delta x = 0.01x$
Since approximately change in surface area $=\Delta S \sim dS = \frac{dS}{dx} dx$
$= \frac{dS}{dx} \Delta x \sim 12x(-0.01x) \sim -0.12x^2$ Square meters (decreasing)

Question6

If the radius of a sphere is measured as 7 m with an error of 0.02 m, then find the approximate error in calculating its volume.

Solution:

Consider r be the radius of the sphere and Δr be the error.
Then, as per question, r =7 m and $\Delta r = 0.02$ m
We know that, Volume of sphere (V) $=\frac{4}{3}\pi r^3$
$\frac{dV}{dr} = \frac{4}{3}\pi . 3r^2$
Approximate error in calculating the volume = Approximate value of ΔV
$dV = \frac{dV}{dr}\ (dr)$
$= \left(\frac{4}{3}\pi 3r^2\right) dr$
$= 4\pi(7)^2\ (0.02)$
$=3.92 \times \frac{22}{7}$
=12.32 m3
Therefore, the approximate error in calculating volume ix 12.32 m3

Question7

If the radius of a sphere is measured as 9 m with an error of 0.03 m, then find the approximate error in calculating its surface area.

Solution:

Consider r be the radius of the sphere.
And, surface area of the sphere (S) = $4\pi r^2$ (formula for SA)
$\frac{dS}{dr} = 8\pi r$
$ds = 8\pi r dr$
$ds = 8\pi r\ \Delta r$
$ds = 8\pi\ (9)(0.3)$
$= 2.16\pi$ Square meters

Question8

If $f(x) = 3x^2 15x + 5$ then the approximate value of f (3.02) is:
(a) 47.66 (b) 57.66 (c) 67.66 (d) 77.66

Solution:

Option (D) correct.

Explanation:
Consider, $f'(x) = \frac{dy}{dx} = y = 3x^2 + 15x + 5$ …………….. (1)
On differentiating equation (1) w.r.t. x, we get
$$f'(x) = \frac{dy}{dx} = 6x + 15$$
Or $dy = (6x + 15)dx = (6x + 15)\Delta x$ ………………. (2)
Changing x to $x + \Delta x$ and y to $y + \Delta y$ in equation (1)
$y + \Delta y = f(x + \Delta x)$
$= f(3.02) = f(3 + 0.02)$ ……………… (3)
Here, $x = 3$ and $\Delta x = 0.02$
So,
From equation (3), $f(3.02) = y + \Delta y$
Since, Δx and Δy is approximately equal todx and dy respectively.
From equation (1) and (2),
$f(3.20) = (3x^2 + 15x + 5) + (6x + 15)\Delta x$
$f(3.02) = 3(9) + 15(3) + 5 + (6 \times 3 + 15)(0.02)$
= 77+0.66 = 77.66

Question9

The approximate change in the volume of a cube of side x meters caused by increasing the side by 3% is:
$0.06\ x^3\ m^3$ (b) $0.6\ x^3\ m^3$ (c) $0.09\ x^3\ m^3$ (d) $0.9\ x^3\ m^3$

Solution:

Option (C) is correct.
Explanation:
We know that, Volume (V) = x^3 …………… (1)
$\frac{dV}{dx} = 3x^2$ ……………. (2)
As there is increase in side = 3% $= \frac{3x}{100}$
We have, $\Delta x = \frac{3x}{100}$ …………….. (3)
Since approximate change in volume V of cube = $\Delta V \sim dV = \frac{dV}{dx}\ dx$
$$= \frac{dV}{dx}\ \Delta x$$
$\sim 3x^2 \left(\frac{3x}{100}\right) \sim \frac{9}{100} x^3 \sim 0.09x^3$ Cubic meters

Exercise6.5

Question1

Find the maximum and minimum values, if any, of the following function given by:
(a) $f(x) = (2x-1)^2 + 3$
(b) $f(x) = 9x^2 + 12x + 2$
(c) $f(x) = -(x-1)^2 + 10$
(d) $g(x) = x^3 + 1$

Solution:

(i) Given function is:
$f(x) = (2x-1)^2 + 3$
As, $(2x-1)^2 \geq 0$ for all $x \in R$
Adding 3 both sides, we get
$(2x-1)^2 + 3 \geq 0 + 3$
$f(x) \geq 3$
The minimum value of f(x) is 3 when 2x-1=0, which means $x = \frac{1}{2}$
The function does not have a maximum value.

(ii) Given function is: $f(x) = 9x^3 + 12x + 2$
Using squaring method for a quadratic equation:
$f(x) = 9\left(x^2 + \frac{4x}{3} + \frac{2}{9}\right)$
$f(x) = 9\left(x^2 + \frac{4x}{3} + \left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^2 + \frac{2}{9}\right)$
$= 9\left[\left(x + \frac{2}{3}\right)^2 - \frac{4}{9} + \frac{2}{9}\right]$
$f(x) = 9\left(x + \frac{2}{3}\right)^2 - 2$ ……….. (1)
As $9\left(x + \frac{2}{3}\right)^2 \geq 0$ for all $x \in R$
Subtracting 2 from both sides,
$9\left(x\frac{2}{3}\right)^2 \geq -0 - 2$
Therefore, minimum value of f(x) is -2and is obtained when
$x + \frac{2}{3} = 0$, that, $x = \frac{-2}{3}$
And this function does not have a maximum value.

(iii) Given function is: $f(x) = -(x-2)^2 + 10$ ……….. (1)
As $(x-1)^2 \geq 0$ forallx $\in R$
Multiplying both sides by -1 and adding 10 both sides,
$-(x-1)^2 + 10 \leq 10$
$f(x) \leq 10$ [Using equation (1)]

Maximum value of f(x) is 10 which is obtained when
X – 1 = 0 which implies x=1.
And minimum value of f(x) does not exit.

(iv) Given function is:$g(x) = x^3 + 1$
At $x \to \infty$ $g(x) \to \infty$
At $x \to -\infty$ $g(x) \to \infty$
Hence, maximum value and minimum value of g(x) do not exit.

Question2

Find the maximum and minimum values, if any, of the following functions given by:

i. $f(x) = |x + 2| - 1$
ii. $g(x) = |x + 1| - 1$
iii. $h(x) = \sin(2x) + 5$
iv. $f(x) = |\sin 4x + 3|$
v. $h(x) = x + 1, x \in (-1,1)$

Solution:

(i) Given function is: $f(x) = |x + 2| - 1$ .... (1)
As $|x + 2| \geq 0$ for all $x \in R$
Subtracting 1 from both sides, $|x + 2| - 2 \geq -1$
$f(x) \geq -1$
Therefore minimum value of f(x) is- 1 which is obtained when x +2=0 or x=-2.
From equation (1), maximum value of $f(x) \to \infty$ hence it does not exit.

(ii) Given function is:$g(x) = -|x + 1| + 3$
As $|x + 1| \geq 0$ for all $x \in R$
Multiplying by – 1 both sides and adding 3 both sides,
$-|x + 1| + 3 \leq 3$
$g(x) \leq 3$
Maximum value of g (x) is 3 which is obtained when x+1=0 or x=-1.
From equation (1), minimum value of$g(x) \to \infty$, does not exit.

(iii) Given function is: $h(x) = \sin(2x) + 5$ .... (1)
As -1$\leq \sin 2x \leq 1$ for all $x \in R$
Adding 5 to all sides, $-1 + 5 \leq \sin 2x + 5 \leq 1 + 5$
$4 \leq h(x) \leq 6$
Therefore, minimum value of h(x) is 4 and maximum value is 6.

(iv) Given function is: $f(x = |\sin 4x + 3|$
As $-1 \leq \sin 4x \leq 1$ for all $x \in R$
Adding 3 to all sides, $-1 + 3 \leq \sin 2x + 5 \leq 1 + 3$
$2 \leq f(x) \leq 4$

Therefore, minimum value of f(x) is 2 and maximum value is 4.
(v) Given function is:$h(x) = x + 1, x \in (-1,1)$ …………. (1)
As $-1 < x < 1$
Adding 1 both sides, $-1 + 1 < x + 1 < 1 + 1$
$0 < h(x) < 2$
Therefore, neither minimum value not maximum value of h(x) exists.

Question3

Find the local maxima and local minima, if any, of the following functions. Find also the local maximum and the local minimum values, as the case may be:

i. $f(x) = x^2$
ii. $g(x) = x^{3-}\ 3x$
iii. $h(x) = \sin x + \cos x, 0 < x < \frac{\pi}{2}$
iv. $h(x) = \sin x - \cos x, 0 < x < 2\pi$
v. $f(x) = x^3 - 6x^2 + 9x + 15$
vi. $g(x)\frac{x}{2} + \frac{2}{x}, x > 0$
vii. $g(x) = \frac{1}{x^2+2}$
viii. $f(x) = x\sqrt{1 - x, x > 0}$

Solution:

(i) Given function is: $f(x) = x^2$
$f^{'}(x) = 2x$ and $f''(x) = 2$
Now $f^{'}(x) = 0$
X=0 [Turning point]
Again, when x=0, $f''(x) = 2$ [Positive]
Therefore, x=0, is a point of local minima and local minimum value=$f(0) = (0)^2 = 0$

(ii) Given function is: $g(x) = x^3 - 3x$
$g^{'}(x) = 3x^2 - 3$ and $g''(x) = 6x$
Now $g^{'}(x) = 0$
$3x^2 - 3 = 0$
$3(x^2 - 1) = 0$
$3(x + 1)(x - 1) = 0$
$x = -1$ or $x = 1$ [Turning points]
Again when$x = -1$,
$g''(x) = 6x = 6(-1)$[Negative]
$x = -1$ Is a point of local maxima and local maximum value? $g(-1) = (-1)^3 - 3(-1) = 2$
And
When x=1; $g''(x) = 6x = 6(1) = 6$ [Positive]
x =1, is a point of local minima and local minimum value $g(1) = (1)^3(1) = 2$

(iii) Given function is: $h(x) = \sin x + \cos x \left(0 < x < \frac{\pi}{2}\right)$ ………… (1)

$h'(x) = \cos x - \sin x$ and $h''(x) = -\sin x - \cos x$

Now $h'(x) = 0$

$\cos x - \sin x = 0$

$-\sin x = -\cos x$

$\frac{\sin x}{\cos x} = 1$

$\tan x = 1$ [Positive]

X can have values in both 1st and 3rd quadrant.

But, $\left(0 < x < \frac{\pi}{2}\right)$ therefore, x is only in I quadrant.

As, $\tan x = 1 \frac{\pi}{4}$

At $x = \frac{\pi}{4} h''(x) = -\sin x - \cos x$

$= h''(x) = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4}$

$= \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2}$ [Negative]

$x = \frac{\pi}{4}$ is a point of local maxima and local maximum value

$= h\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$

(iv) Given function is: $f(x) = \sin x - \cos x (0 < x < 2\pi)$ …………… (1)

$f'(x) = \cos x + \cos x$ And $f''(x) = \sin x + \cos x$

Now $f'(x) = 0$

$\cos x + \sin x = 0$

$\frac{\sin x}{\cos x} = -1$

$\tan x = -1$ [Negative]

X can have values in both 2nd and 4th quadrant.

$\tan x = -1 = -\tan\frac{\pi}{4}$

$\tan\left(\pi - \frac{\pi}{4}\right)$ Or $\tan\left(2\pi - \frac{\pi}{4}\right)$

$\tan x = \tan\frac{3\pi}{4}$ or $\tan\frac{7\pi}{4}$

$x = \frac{3\pi}{4}$ and

$x = \frac{7\pi}{4}$

At $x = \frac{3\pi}{4} f''(x) = -\sin x + \cos x = -\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4}$

$= h''(x) = -\sin\left(\pi - \frac{\pi}{4}\right) + \cos\left(\pi - \frac{\pi}{4}\right)$

$= = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4}$

$= \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$

$= \frac{-2}{\sqrt{2}} = -\sqrt{2}$ [Negative]

So, $x = \frac{3\pi}{4}$ is a point of local maxima and local maximum value= $f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4}$

$= \sin\left(\pi - \frac{\pi}{4}\right) - \cos\left(\pi - \frac{\pi}{4}\right)$

$= \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$

At $x = \frac{7\pi}{4}$ $f''(x) = -\sin x + \cos x$

$= -\sin\frac{7\pi}{4} + \cos\frac{7\pi}{4}$

Which implies; $h''(x) = -\sin\left(2\pi - \frac{\pi}{4}\right) + \cos\left(2\pi - \frac{\pi}{4}\right)$

$= \sin\frac{\pi}{4} + \cos\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$

$= \frac{2}{\sqrt{2}} = \sqrt{2}$ [Positive]

$x = \frac{7\pi}{4}$ Is a point of local maxima and maximum value=$f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4}$

$= \sin\left(2\pi - \frac{\pi}{4}\right) - \cos\left(2\pi - \frac{\pi}{4}\right)$

$= -\sin\frac{\pi}{4} - \cos\frac{\pi}{4} = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$

(v) Given function is: $f(x) = x^3 - 6x^2 + 9x + 15$

$f'(x) = 3x^2 - 12x + 9$ and $f''(x) = 6x - 12$

Now $f'(x) = 0$

$3x^2 - 12x + 9 = 0$

$x^2 - 4x + 3 = 0$

$(x-1)(x-3) = 0$

$x = 1$ or $x = 3$ [Turning points]

At $x = 1, f''(x) = 6x - 12 = 6 - 12 = -6$[Negative]

$x = 1$is a point of local minima and local minimum value is $f(1) = (1)^3 - 6(1)^2 + 9(1) + 15 = 19$

At $x = 3, f''(x) = 6x - 2 = 6 \times 3 - 12 = 6$ [Positive]

$x = 3$Is point of local minima and local minimum values is $f(3) = (3)^3 - 6(3)^2 + 9(1) + 15 = 15$

(vi) Given function is: $g(x) = \frac{x}{2} + \frac{2}{x}, x > 0$

$g'(x) = \frac{1}{2} - \frac{2}{x^2}$

$= \frac{x^2 - 4}{2x^2}$

$= \frac{(x+2)(x-2)}{2x^2}andg''(x) = \frac{4}{x^3}$

Now $g'(x) = 0$

$= \frac{(x+2)(x-2)}{2x^2} = 0$

$(x+2)(x-2)=0$
$x=-2$ or $x=2$
But $x>0$, therefore $x=2$ is only the turning point.
$x=2$ Is point of local minima and local minimum value is $g(2)=\frac{2}{2}+\frac{2}{2}=2$

(vii) Given function is: $h(x)=\frac{1}{x^2+2}=(x^2+2)^{-1}$
$h'(x)=(-1)(x^2+2)^{-2}(2x)=\frac{-2x}{(x^2+2)^2}$ and
$$h''(x)=\left[\frac{(x^2+2)^2 2-2x(x^2+2)2x}{(x^2+2)^4}\right]$$
$$=\frac{-2(2-3x^2)}{(x^2+2)^3}$$
Now $h'(x)=0$
$$\frac{-2x}{(x^2+2)^2}=0$$
As, $x=0$
At $x=0, h''(x)=\frac{-2(2-3x^2)}{(x^2+2)^3}=\frac{-2(2-0)}{(0+2)^3}=\frac{-4}{8}=\frac{1}{2}$ [Negative]
$x=0$ Is a point of local maxima and local maximum value is $h(0)=\frac{1}{0+2}=\frac{1}{2}$

(viii) Given function is: $f(x)=x\sqrt{1-x}, x\le 1$
$$f'(x)=x\frac{1}{2}(1-x)^{\frac{-1}{2}}\frac{d}{dx}(1-x)+\sqrt{1-x}.1$$
$$=\frac{-x}{2\sqrt{1-x}}+\sqrt{1-x}$$
$$=\frac{-x+2(1-x)}{2\sqrt{1-x}}=\frac{2-3x}{2\sqrt{1-x}}$$
And $f''(x)=\frac{1}{2}\frac{\sqrt{1-x}.(3)-(2-3x)\frac{1}{2\sqrt{1-x}}(-1)}{1-x}$
$$=\frac{-6(1-x)+2-3x}{4(1-x)^{\frac{3}{2}}}$$
$$=\frac{3x-4}{4(1-x)^{\frac{3}{2}}}$$
Now $f'(x)=0$
$$\frac{2-3x}{2\sqrt{1-x}}=0$$
$2-3x=0$
$$x=\frac{2}{3}$$
$x=\frac{2}{3}$ Is a point of local maxima and local maximum value is

$f\left(\frac{2}{3}\right) = x\sqrt{1-x} = \frac{2}{3}\sqrt{1-\frac{2}{3}} = \frac{2\sqrt{3}}{9}$

$f''\left(\frac{2}{3}\right) = \frac{3\left(\frac{2}{3}\right)-4}{4\left(1-\frac{2}{3}\right)^{\frac{3}{2}}}$

Again

$\frac{2-4}{4\left(\frac{1}{3}\right)^{\frac{3}{2}}} = \frac{-1}{2\left(\frac{1}{3}\right)^{\frac{3}{2}}} < 0$

Therefore, f(x) has local maxima value at $x = \frac{2}{3}$

Question4

Prove that the following function do not have maxima or minima:

i. $f(x) = e^x$
ii. $g(x) = \log x$
iii. $h(x) = x^3 + x^2 + x + 1$

Solution:

(i) Given function is:$f(x) = e^x$

$f'(x) = e^x$

Now $f'(x) = 0$

$e^x = 0$

But this gives no real value of x. Therefore, there is no turning point.

f(x) Does not have maxima or minima.

(ii) Given function is: $g(x) = \log x$

$g'(x) = \frac{1}{x}$

Now $g'(x) = 0$

$\frac{1}{x} = 0$

1=0 (which is not possible)

F(x) does not have maxima and minima.

(iii) Given function is: $h'(x) = x^3 + x^2 + x + 1$

$h'(x) = 3x^2 + 2x + 1$

Now $h'(x) = 0$

$3x^2 + 2x + 1 = 0$

$= \frac{-2 \pm \sqrt{4-12}}{6}$

$= \frac{-2 \pm \sqrt{-8}}{6}$

$= \frac{-1 \pm \sqrt{2}i}{3}$

Here, values of x are imaginary.

H(x) does not have maxima or minima.

Question5

Find the absolute maxima value and the absolute minimum value of the following functions in the given intervals:

i. $f(x) = x^3, x \in [-2,2]$

ii. $f(x) = \sin x + \cos x, x \in [0, \pi]$

iii. $f(x) = 4x - \frac{1}{2}x^2, x \in \left[-2, \frac{9}{2}\right]$

iv. $f(x) = (x-1)^2 + 3, x \in [-3,1]$

Solution:

(i) Given function is: $f(x) = x^3, x \in [-2,2]$

$f'(x) = 3x^2$

Now $f'(x) = 0$

$3x^2 = 0$

$x = 0 \in [-2,2]$

At $x = 0, f(0) = 0$

At $x = -2, f(-2) = (-2)^3 = -8$

At $x = 2, f(2) = (2)^3 = 8$

Therefore, absolute minimum value of f(x)is − 8 and absolute maximum value is 8.

(ii) Given function is: $f(x) = \sin x + \cos x, x \in [0, \pi]$

$f'(x) = \cos x - \sin x$

Now $f'(x) = 0$

$\cos x - \sin x = 0$

$-\sin x = -\cos x$

$\tan x = 1$ [Positive]

X lies in I quadrant.

$\tan x = 1 = \tan\frac{\pi}{4}$

So, $x = \frac{\pi}{4}$

$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$

$f(0) = \sin 0 + \cos 0 = 0 + 1 = 1$

$f(0) = \sin\pi + \cos\pi = 0 - 1 = -1$

Therefore, absolute minimum value is-1 and absolute maximum value is 1.

(iii) Given function is: $f(x) = 4x - \frac{1}{2}x^2, x \in \left(-2\frac{9}{2}\right)$

$f(x) = 4 - \frac{1}{2}(2x) = 4 - x$

Now $f^{'}(x) = 0$

$4 - x = 0$

$x = 4 \in \left(-2, \frac{9}{2}\right)$

At $x = 4, f(4) = 16 - \frac{1}{2}(16) = 16 - 8 = 8$

Atx $= -2, f(-2) = 4(-2) - \frac{1}{2}(4) = -8 - 2 = -10$

At $x = \frac{9}{2}, f\left(\frac{9}{2}\right) = 4\left(\frac{9}{2}\right) - \frac{1}{2}\left(\frac{9}{2}\right)^2 = 18 - \frac{81}{8} = \frac{63}{8}$

Therefore, absolute minimum value is -10 and absolute maxima value is 8.

(iv) Given function is: $f(x) = (x - 1)^2 + 3, x \in (-3,1)$

$f^{'}(x) = 2(x - 1)$

Now $f^{'}(x) = 0$

$2(x - 1) = 0$

$x = 1 \in (-3,1)$

At $x = 1, f(1) = (1 - 1)^2 + 3 = 3$

At $x = -3, f(-3) = (-3 - 1)^2 + 316 + 3 = 19$

Therefore, absolute minimum value is 3 and absolute maximum value is 19.

Question6

Find the maximum profit that a company can make, if the profit function is given by $p(x) = 31 + 24x - 18x^2$

Solution:

Given function is: Profit function $p(x) = 31 + 24x - 18x^2$

$p^{'}(x) = 24 - 36x$ and $p''(x) = -36$

Now $p^{'}(x) = 0$

$24 - 36x = 0$

$x = \frac{24}{36} = \frac{2}{3}$

At $x = \frac{2}{3}, p''(x) = -36$ [Negative]

$p(x)$ Has a local maximum value at $x = \frac{2}{3}$

Atx $= \frac{2}{3}$, Maximum profit

$= 41 + 24\left(\frac{2}{3}\right) - 18\left(\frac{4}{9}\right)$

=41+16-8=49

Question7

Find both the maximum value and minimum value of $3x^4 - 8x^3 + 12x^2 - 48x + 25$ on the interval $[0,3]$.

Solution:

Consider $f(x)3x^4 - 8x^3 + 12x^2 - 48x + 25 \text{on} [0, 3]$
$f'(x) = 12x^3 - 24x^2 + 24x - 48$
Now $f'(x) = 0$
$12x^3 - 24x^2 + 24x - 48 = 0$
$x^3 - 2x^2 + 2x - 4 = 0$
$(x - 2)(x^2 + 2) = 0$
$x = 2 \text{ or } x = \pm\sqrt{2}$
As $x = \pm\sqrt{2}$ is imaginary, therefore it is rejected.
Here $x = 2$ is turning point.
At $x = 2, f(2) = 3(16) - 8(8) + 12(4) - 48(2) + 25 = -39$
At $x = 0$ $f(0) = 25$
At $x = 3, f(3) = 3(81) - 8(27) + 12(9) - 48(3) + 25 = 16$
Therefore, absolute minimum value is -39 and absolute maximum value is 25.

Question8

At what points on the interval? $[0, 2\pi]$ Does the function sin 2x attain its maximum value?

Solution:

Consider $f(x) = \sin 2x$
$f'(x) = 2\cos 2x$
Now $f'(x) = 0$
$2\cos 2x = 0$
$2x = (2n + 1)\frac{\pi}{2}$
$x = (2n + 1)\frac{\pi}{4}$
Put $n = 0,1,2,3, x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \in [0,2\pi]$
Now $f(x) = \sin 2x$
$f\left[(2n + 1)\frac{\pi}{4}\right] = \sin(2n + 1)\frac{\pi}{2}$
$= \sin\left(n\pi + \frac{\pi}{2}\right)$
$= (-1)^n \sin\frac{\pi}{2} = (-1)^n$
Putting $n = 0,1,2,3;$
$f\left(\frac{\pi}{4}\right) = (-1)^0 = 1$

$f\left(\frac{3\pi}{4}\right) = (-1)^1 = -1$
$f\left(\frac{5\pi}{4}\right) = (-1)^2 = 1$
$f\left(\frac{7\pi}{4}\right) = (-1)^3 = -1$
Also $f(0) = \sin 0 = 0$ and $f(2\pi) = \sin 4\pi = 0$
As $f(x)$ attains its maximum value 1 at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$
Therefore, the required points are $\left(\frac{\pi}{4}, 1\right)$ and $\left(\frac{5\pi}{4}, 1\right)$

Question9

What is the maximum value of the function $\sin x + \cos x$?

Solution:

Consider $f(x) = \cos x - \sin x$
$f'(x) = \cos x - \sin x$
Now $f'(x) = 0$
$\cos x - \sin x = 0$
$-\sin x = -\cos x$
$\tan x = 1 = \tan\frac{\pi}{4}$
Here $x = n\pi + \frac{\pi}{4}$ is a turning point.
$f\left(n\pi + \frac{\pi}{4}\right) = \sin\left(n\pi + \frac{\pi}{4}\right) + \cos\left(n\pi + \frac{\pi}{4}\right)$
$= (-1)^n \frac{1}{\sqrt{2}} + (-1)^n \cos\frac{\pi}{4}$
$= (-1)^n \frac{1}{\sqrt{2}} + (-1)^n \frac{1}{\sqrt{2}}$
$= 2(-1)^n \frac{1}{\sqrt{2}}$
$= \sqrt{2}(-1)^n$
If n is even, then $f\left(n\pi + \frac{\pi}{4}\right) = \sqrt{2}$
If n is odd, then $f\left(n\pi + \frac{\pi}{4}\right) = -\sqrt{2}$
Therefore, maximum value of $f(x)$ is $\sqrt{2}$ and minimum value of $f(x)$ is $-\sqrt{2}$

Quetion10

Find the maximum value of $2x^2 - 24x + 107$ in the interval [1, 3]. Find the maximum value of the same function in $[-3, -1]$

Solution:

Consider $f(x) = 2x^3 - 24x + 107$
$f'(x) = 6x^2 - 24$
Now $f'(x) = 0$
$6x^2 - 24 = 0$
$x^2 = 4$
$x = \pm 2$
$x = 2$ or$x = -2$ [Turning points]
For Interval $[1, 3]$, $x = 2$ is turning point.
At $x = 1, f(1) = 2(1) - 24(1) + 107 = 85$
At $x = 2, f(2) = 2(8) - 24(2) + 107 = 75$
At $x = 3, f(3) = 2(27) - 24(3) + 107 = 89$
There, maximum value of $f(x)$ is 89.
For Interval $[-3, -1]$, $x = -2$ is turning point.
At $x = -1, f(1) = 2(-1) - 24(-1) + 107 = 129$
At $x = -2, f(2) = 2(-8) - 24(-2) + 107 = 139$
At $x = -3$ $f(3) = 2(-27) - 24(-3) + 107 = 125$
Therefore, maximum value of $f(x)$is 139.

Question11

It is given that at$x = 1$, the function $x^4 - 62x^2 + ax + 9$attains its maximum value, on the interval $[0, 20]$. Find the value of$a$.

Solution:
Consider $f(x) = x^4 - 62x^2 + ax + 9$
$f'(x) = 4x^2 - 124x + a$
As, $f(x)$attains its maximum value at$x = 1$ in the interval$[0, 2]$, therfor$f'(1) = 0$
$f'(1) = 4 - 124 + a = 0$
$a - 120 = 0$
$a = 120$

Question12

Find the maximum and minimum value of $x + \sin x$ on $[0, 2\pi]$

Solution:

Consider $f(x) = x + \sin 2x$
$f'(x) = 1 + 2\cos 2x$
Now $f'(x) = 0$
$1 + 2\cos 2x = 0$
$2\cos 2x = -1$

$\cos 2x = \frac{-1}{2}$
$= -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right)$
$= \cos\frac{2\pi}{3}$
$2x = 2n\pi \pm \frac{2\pi}{3}$ Where $n \in \mathbf{Z} = x = n\pi \pm \frac{n\pi}{3}$
For $n = 0, x = \pm\frac{\pi}{3}$ But $x = -\frac{\pi}{3} \nexists [0,2\pi]$, therefore $x = \frac{\pi}{3}$
For $n = 1, x = \pi \pm \frac{\pi}{3} = \pi + \frac{\pi}{3}$ and $\pi - \frac{\pi}{3}$
Forn $= 2, x = 2\pi \pm \frac{\pi}{3}$
But $x = 2\pi + \frac{\pi}{3} > 2\pi \nexists [0,2\pi]$ therfore $x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$
Therefore, it is clear that the only turning point of f(x)given by$x + \sin 2x$ which belong to given closed interval $[0, 2\pi]$ are, $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$
At $x = \frac{\pi}{3}$
$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2} = 1.05 + 0.87 = 1.92$(approx.)
At $x = \frac{2\pi}{3}$
$f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \sin\frac{4\pi}{3} = 2\pi - \frac{\sqrt{3}}{2} = 2.10 - 0.87 = 1.23$ (approx.)
At $x = \frac{4\pi}{3}$
$f\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} + \sin\frac{8\pi}{3} = \frac{4\pi}{3} + \frac{\sqrt{3}}{2} = 4 \times 1.05 + 0.87 = 5.07$ (approx.)
At $x = \frac{5\pi}{3}$
$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sin\frac{10\pi}{3} = \frac{5\pi}{3} - \frac{\sqrt{3}}{2} = 5 \times 1.05 - 0.87 = 4.38$ (approx.)
At $x = 0$ $f(0) = 0 + \sin 0 = 0$
At $x = 2\pi$
$f(2\pi) = 2\pi + \sin 4\pi = 2\pi + 0 = 2\pi = 2 \times 3.14 = 6.28$(approx.)
Therefore, Maxima value = 2π and
Minimum value = 0

Question13

Find two numbers whose sum is 24 and whose product is as large as possible.

Solution:

Consider the two numbers be x and y
According to the question, $x + y = 24$
$y = 24 - x$ .... (i)
And Considerz is the product of x and y.
$z = xy$

$z = x(24 - x)$ [From equation (i)]
$z = 24x - x^2$
$\frac{dz}{dx} = 24 - 2x$ And $\frac{d^2z}{dx^2} = -2$
Now to find turning point, $\frac{dz}{dx} = 0$
$24 - 2x = 0 \Rightarrow x = 12$
At $x = 12, \frac{d^2z}{dx^2} = -2$ [Negative]
$x = 12$ Is a point of local maxima and z is maximum at $x = 12$
From equation (i) $y = 24 - 12 = 12$
Therefore, the two required numbers are 12 and 12.

Question14

Find two positive integers x and y such that $x + y = 60$ and xy^3 is maximum.

Solution:

Given function is: $x + y = 60, x > 0, y > 0$ .... (i)
Consider P= xy^3 [To be maximized] .... (ii)
Putting from equation (i), $x = 60 - y$ in equation (ii),
$P = (60 - y)y^3 = 60y^3 - y^4$
$\frac{dP}{dy} = 180y^2 - 4y^3 = 4y^2(45 - y)$ .... (iii)
Now $\frac{dP}{dy} = 0$

$4y^2(45 - y) = 0$
Y = 0, 45
It is clear that $\frac{dP}{dy}$ changes sign from positive to negative as y increases through 45.
Therefore, P is maximum when $y = 45$
Hence, xy^3 is maximum when $x = 60 - 45 = 15$ and $y = 45$

Question15

Find two positive integers x and y such that their sum is 35 and the product x^2y^5 is a maximum.

Solution:

Given function is: $x + y = 35$
$y = 35 - x$ .... (i)
Consider $z = x^2y^5$
$x^2(35 - x)^5$ [From equation (i)]

$$\frac{dz}{dx} = x^2.5(35-x)^4(-1) + (35-x)^5 2x$$
$$\frac{dz}{dx} = x(35-x)^4[-5x + (35-x)2]$$
$$\frac{dz}{dx} = x(35-x)^4[-5x + 70 - 2x]$$
$$\frac{dz}{dx} = x(35-x)^4(70-7x)$$
$\frac{dz}{dx} = 7x(35-x)^4(10-x)$ .... (ii)

Now $\frac{dz}{dx} = 0$

$7x(35-x)^4(10-x) = 0$

$x = 0$ or $35 - \ x = 0$ or $10 - x = 0$

$x = 0$ or $x = 35$ or $x = 10$

Now $x = 0$ is rejected because according to question, x is a positive number.

Also $x = 35$ is rejected because from equation (i), $y = 35 - 35 = 0$, but y is positive.

Therefore, x=10 is

Only the turning point.

$$\frac{d^2z}{dx^2} = 7(35-x)^3(6x^2 - 120x + 350)$$

At $x = 10, \frac{d^2z}{dx^2} = 7(35-10)^3(6 \times 100 - 120 \times 10 + 350)$

$= 7(25)^3(-250) < 0$

By second derivative test, $\frac{dz}{dx}$ will be maximum at $x = 10$ when $y = 35 - 10 = 25$

Therefore, the requires numbers are 10 and 25.

Question16

Find two positive integers whose sum is 16 and sum of whose cubes is minimum.

Solution:

Consider the two positive numbers are x and y

$x + y = 16$

$y = 16 - x$ .... (i)

Consider $z = x^3 + y^3$

$z = x^3 + (16-x)^3$ [From equation (i)]

$z = x^3 + (16)^3 - x^3 - 48x(16-x)$

$= (16)^3 - 768x + 48x^2$

$\frac{dz}{dx} = -768 + 96x$ and $\frac{d^2z}{dx^2}$ 69

Now $\frac{dz}{dx} = 0$

$-768 + 96x = 0$

$x = 8$

At $x = 8\,\frac{d^2z}{dx^2} = 96$ is positive.

$x = 8$ Is a point of local minima and z is minimum when $x = 8$

$y = 16 - 8 = 8$

Therefore, the required numbers are 8 and 8

Question17

A square piece of tin of side 18 cm is to be made into a box without top, by cutting a square from each corner and folding up the flaps to from the box. What should be the side of the square to be cut of so that the volume of the box is the maximum possible?

Solution:

Each side of square piece of tin is 18 cm.

Consider x cm be the side of each of the four squares cut off from each corner.

Then dimensions of the open box formed by folding the flaps after cutting off squares are (18-2x), (18-2x) and x cm.

Consider z denotes the volume of the open box.

$z = (18 - 2x), (18 - 2x)x$

$z = (18 - 2x)^2x$

$z = (324 + 4x^2 72x)x$

$= 4x^2 - 72x^2 + 324x$

Which implies

$\frac{dz}{dx} = 12x^2 - 144x + 324$ and $\frac{d^2z}{dx^2} = 24x - 144$

Now $\frac{dz}{dx} = 0$

$12x^2 - 144 + 324 = 0$

$= x^2 - 12x + 27 = 0$

$(x - 9)(x - 3) = 0$

$x = 9$ or $x = 3$

$x = 9$is rejected because at $x = 9$ length $=18 - 2x = 18 - 2 \times 9 = 0$ which is impossible

$x = 3$is the turning point.

At $x = 3, \frac{d^2z}{dx^2} = 24 \times 3 - 144 = -72$ [Negative]

Z is minimum at x=3 that is, side of each square to be cut of from each corner for maximum volume is 3 cm.

Question18

A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be side of the square to be cut off so that the volume of the box is maximum?

Solution:

Length and breadth of a rectangular sheet is 45 cm and 24 cm respectively.
Consider x cm be the side of each of the four squares cut off from each corner.
Then dimensions of the open box formed by folding the flaps after cutting off squares are $(45 - 2x, (24-2x)$ and x cm
Consider z denotes the of the open box
$z = (45 - 2x)(24 - x)x$
$z = (1080 - 138x + 4x^2)x$
$= 4x^3 - 138^2 + 1080x$
$\frac{dz}{dx} = 12x^2 - 276x + 1080$ and $\frac{d^2z}{dx^2} = -24x - 276$
Now $\frac{dz}{dx} = 0$
$12x^2 - 276x + 1080 = 0$
$= x^2 - 23x + 90 = 0$
$(x - 5)(x - 18) = 0$
$x = 5$ or $x = 18$
$x = 18$ is rejected because at $x = 18$ length $= 24 - 2x = 18 - 2 \times 18 = -12$ which is impossible.
Here x=5 is the turning point.
At $x = 5, \frac{d^2z}{dx^2} = 24 \times 3 - 276 = -156$ [Negative]
Z is minimum at x=5 that is, side of each square to be cut off from each corner for maximum volume is 5 cm.

Question19

Show that all of the rectangles inscribed in a given fixed circle, the square has maximum area.

Solution:

Consider PQRS be the rectangle in a given circle with centre O and radius a Consider x and y be the length and breadth of the rectangle, that is> 0 *and* $y > 0$
In right angled triangle PQR, using Pythagoras theorem,
$PQ^2+QR^2=PR^2$

$x^2 + y^2 = (2a)^2$
$y^2 = 4a^2 - x^2$
$y = \sqrt{4a^2 - x^2}$ .... (1)
Consider A be the area of the rectangle, then a= xy= $x\sqrt{4a^2 - x^2}$
$$\frac{dA}{dx} = \sqrt{4a^2 - x^2} + x\frac{1}{2\sqrt{4a^2 - x^2}}(-2) = \sqrt{4a^2 - x^2} - \frac{x^2}{\sqrt{4a^2 - x^2}}$$
$$= \frac{4a^2 - 2x^2}{\sqrt{4a^2 - x^2}}$$
$$\frac{d^2A}{dx^2} = \frac{\sqrt{4a^2 - x^2}(-4x) - (4a^2 - 2x^2)\frac{(-2x)}{2\sqrt{4a^2 - x^2}}}{(4a^2 - 2x^2)}$$
And $\frac{(4a^2-2x^2)(-4x)+x(4a^2-2x^2)}{(4a^2-2x^2)^{\frac{3}{2}}}$
$$= \frac{(4a^2-2x^2)(-4x)+x(4a^2-2x^2)}{(4a^2-2x^2)^{\frac{3}{2}}}$$
$$= \frac{d^2a}{dx^2} = \frac{-12a^2x + 2x^3}{(4a^2 - 2x^2)^{\frac{3}{2}}}$$
$$= \frac{-2x(6a^2 - x^2)}{(4a^2 - 2x^2)^{\frac{3}{2}}}$$
Now $\frac{dA}{dx} = 0$
$$\frac{4a^2 - 2x^2}{\sqrt{4a^2 - x^2}} = 0$$
$4a^2 - 2x^2 = 0$
$x = \sqrt{2}a$
At $x = \sqrt{2}a, \frac{d^2A}{dx^2} = \frac{-2(\sqrt{2}a)(6a^2-2a^2)}{2\sqrt{2}a^3} = \frac{-8\sqrt{2}a^3}{2\sqrt{2}a^3} = -4$
At $x = \sqrt{2}a$, area of rectangle is maximum.
And from equation (1),$y = \sqrt{4a^2 - 2a^2} = \sqrt{2}a$,
That is, $x = y = \sqrt{2}a$
Therefore, the area of inscribed rectangle is maximum when it is square.

Question20

Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.

Solution:

Consider x is the radius of the circular base and y be the height of closed right circular cylinder.
Formula for Total surface area (S) = $2\pi xy + 2\pi x^2$
$xy + x^2 = \frac{S}{2\pi} = k$ (Say)
$xy = k - x^2$

$y = \frac{k-x^2}{x}$ ………….. (i)

Volume of cylinder (z) $= \pi x^2 y$

$= \pi x^2 \left(\frac{k-x^2}{x}\right)$ [From equation]

$z = \pi x(k - x^2) = (kx - x^3)$

$\frac{dz}{dx} = \pi(k - 3x^2)$ and

$\frac{d^2z}{dx^2} \pi(-6x) = -6\pi x$

Now $\frac{dz}{dx} = 0$

$\pi(k - 3x^2) = 0$

$x = \sqrt{\frac{k}{3}}$

At $x = \sqrt{\frac{k}{3}} \frac{d^2z}{dx^2} = -6\pi\sqrt{\frac{k}{3}}$ [Negative]

z Is maximum at $x = \sqrt{\frac{k}{3}}$

From equation (1), $y = \frac{k-\frac{k}{3}}{\sqrt{\frac{k}{3}}}$

$= 2\sqrt{\frac{k}{3}} = 2x$

Which implies, Height = Diameter

Therefore, the volume of cylinder is maximum when its height is equal to the diameter of its base.

Question21

Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimeters, find the dimensions of the can which has the minimum surface area.

Solution:

Consider x be the radius of the circular base and y be the height of closed right circular cylinder.

According to the question, Volume of the cylinder $\pi x^2 y = 100$

$\mathbf{y} = \frac{100}{\pi x^2}$ ………… (i)

Total surface area (S) $= 2\pi xy + 2\pi x^2$

$= 2\pi(xy + x^2)$

$= 2\pi\left(x\frac{100}{\pi x^2} + x^2\right)$ [From equation (1)

$S = 2\pi\left(\frac{100}{\pi x} + x^2\right)$

$= 2\pi\left(\frac{100}{\pi}x^{-1} + x^2\right)$

$\frac{dS}{dx} = 2\pi\left(-\frac{100}{\pi}x^{-2} + 2x\right)$ and

$\frac{d^2S}{dx^2} = 2\pi\left(\frac{200}{\pi}x^{-3} + 2\right)$

Now $\frac{dS}{dx} = 0$

$2\pi\left(-\frac{100}{\pi}x^{-2} + 2x\right) = 0$

$\left(-\frac{100}{\pi}x^{-2} + 2x\right) = 0$

$\frac{100}{\pi}x^{-2} = 2x$

$x^3 = \frac{100}{2\pi} = \frac{50}{\pi}$

$x = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$

At $x = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ $\frac{d^2S}{dx^2} = 2\pi\left(\frac{200}{\pi\left(\frac{50}{\pi}\right)} + 2\right)$

$= 2\pi(4 + 2) = 12\pi$ [Positive]

S is minimum when

Radiusx $= \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm

From equation (1)

$y = \frac{100}{\frac{(50)^{\frac{2}{3}}}{\pi}}$

$= 2\left(\frac{50}{\pi}\right)^{\frac{1}{3}} = 2x$

Question22

A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?

Solution:

Consider x meters be the side of square and y meters be the radius of the circle. Length of the wire = Perimeter of square + Circumference of circle

$4x + 2\pi y = 28$

$2x + \pi y = 14$

$y = \frac{14-2x}{\pi}$ ………… (i)

Area of square = $x^2$and Area of circle == πy^2

Combined area (A) = $x^2 + \pi y^2 = x^2 + \pi\left(\frac{14-2x}{\pi}\right)^2$

$= x^2 + \frac{4}{\pi}(7-x)^2$

$\frac{dA}{dx} = 2x - \frac{8}{\pi}(7-x)$ and

$\frac{d^2A}{dx^2} = 2 + \frac{8}{\pi}$

Now $\frac{dA}{dx} = 0$

$2x - \frac{8}{\pi}(7-x) = 0$

$2x - \frac{8}{\pi}(7-x)$

$2\pi x = 56 - 8x$

$(2\pi + 8)x = 56$

$x = \frac{56}{2\pi + 8} = \frac{28}{\pi + 4}$

And $\frac{d^2A}{dx^2} = 2 + \frac{8}{\pi}$ [Positive]

A is minimum when $x = \frac{28}{\pi+4}$

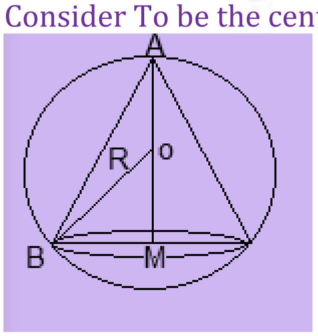
Therefore, the wire should be cut at a distance $4x = \frac{28}{\pi+4}$ from one end.

Question23

Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

Solution:

Consider To be the centre and R be the radius of the given sphere, BM = x and AM = y



In right angled triangle OMB,

OM2 + BM2 = OB2

Using Pythagoras theorem

$(y - R)^2 + x^2 = R^2$

$y^2 + R^2 - 2Ry + x^2 = R^2$
$y^2 - 2Ry + x^2 = 0$
$x^2 = 2Ry - y^2$ …….. (1)
Volume of a cone inscribed in the given sphere
$(z) = \frac{1}{3}\pi x^2 = \frac{1}{3}\pi(2Ry - y^2)y$
$z = \frac{\pi}{3}(2Ry^2 - y^3)$ …………… (2)
$\frac{dz}{dx} = \frac{\pi}{3}(4Ry - 3y^2)$ and $\frac{d^2z}{dx^2} = \frac{\pi}{3}(4R - 6y)$
Now $\frac{dz}{dx} = 0$
$\frac{\pi}{3}(4Ry - 3y^2) = 0$
$4Ry - 3y^2 = 0$
$3y^2 = 4Ry$
$y = \frac{4R}{3}$
At $y = \frac{4R}{3}\frac{d^2z}{dx^2} = \frac{\pi}{3}\left(4R - 6\frac{4R}{3}\right)$
$= \frac{\pi}{3}(4R - 8R)$
$= \frac{-4R}{3}$ [Negative]
z is maximum at $y = \frac{4R}{3}$
Substitute the value of y in equation (1), we get
$x^2 = 2R\frac{4R}{3}\left(\frac{4R}{3}\right)^2 = \frac{8R^2}{3} - \frac{16R^2}{9}$
$= \frac{8R^2}{9}$
Therefore, Maximum volume of the cone
$= \frac{1}{3}\pi x^2 y = \frac{1}{3}\pi\frac{8R^2}{9}\frac{4R}{3} = \frac{8}{27}\frac{4}{3}\pi R^3$
$= \frac{8}{27}$ (Volume of the sphere)

Question24

Show that the right circular cone of least curve surface and given volume has an altitude equal to $\sqrt{2}$ time the radius of the base.

Solution:

Consider x is the radius and y is the height of the cone.
Volume of the cone (V) $= \frac{1}{3}\pi x^2 y$

$x^2y = \frac{3V}{\pi} = k$ (Say) ……………. (1)

And Surface area of the cone (S) $=\pi x\sqrt{x^2 + y^2}$

$S^2 = \pi^2x^2(x^2 + y^2) = z$

$z = \pi^2 \frac{k}{y}\left(\frac{k}{y} + y^2\right)$

$= \pi^2 k\left(\frac{k}{y^2} + y\right)$

$= \pi^2 k(ky^{-2} + y)$

$\frac{dz}{dy} = \pi^2 k[-2k^{-3} + 1]$ and

$\frac{d^2z}{dy^2}\pi^2 k[6ky^{-4}] = \frac{6\pi^2k^2}{y^4}$

Now $\frac{dz}{dy} = 0$

$\pi^2k[-2ky^{-3} + 1] = 0$

$\frac{-2k}{y^3} + 1 = 0$

$\frac{2k}{y^3} = 1$

$y^3 = 2k$

$y = (2k)^{\frac{1}{3}}$ ………… (3)

At $y = (2k)^{\frac{1}{3}}$

$\frac{d^2z}{dy^2} = \frac{6\pi^2k^2}{(2k)^{\frac{4}{3}}}$ [Positive]

z is minimum when $Y = (2K)^{\frac{1}{3}}$

Again, from equation (1), $x^2 = \frac{k}{y} = \frac{k}{(2k)^{\frac{1}{3}}}$

$= \frac{2k}{2(2k)^{\frac{1}{3}}} = \frac{(2k)^{\frac{2}{3}}}{2} = \frac{y^2}{2}$ [From equation (3)]

$y^2 = 2x^2$

$y = \sqrt{2}x$

Therefore, Surface area is minimum when height = $\sqrt{2}$ (radius of base)

Question25

Show that the semi-vertical angle of the cone of the maximum value and of given slant height is $\tan^{-1}\sqrt{2}$

Solution:

Consider x be the radius, y be the height, l be the slant height of given coneθ and be the semi-vertical angle of cone.

$l^2 = x^2 + y^2$

$x^2 = l^2 - y^2$ ……… (1)

Formula for Volume of the cone (V) = $\frac{1}{3}\pi x^2 y$

$V = \frac{1}{3}\pi(l^2 - y^2)y$

$= \frac{\pi}{3}(l^2 y - y^3)$

$\frac{dV}{dy} = \frac{\pi}{3}(l^2 - 3y^2)$ and

$\frac{d^2V}{dy^2} = \frac{\pi}{3}(-6y) = -2\pi y$

Now $\frac{dV}{dy} = 0$

$\frac{\pi}{3}(l^2 - 3y^2) = 0$

$l^2 - 3y^2 = 0$

$3y^2 = l^2$

$y = \frac{1}{\sqrt{3}}$

At $y = \frac{1}{\sqrt{3}} \frac{d^2V}{dy^2} = -2\pi\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{-2\pi l}{\sqrt{3}}$ [Negative]

V is maximum at $y = \frac{1}{\sqrt{3}}$

$x = \sqrt{2}\frac{1}{\sqrt{3}}$

Semi- Vertical angle, $\tan\theta = \frac{x}{y}$

$\frac{\sqrt{2}\frac{l}{\sqrt{3}}}{\frac{l}{\sqrt{3}}} = \sqrt{2}$

Which implies, $\theta = \tan^{-1}\sqrt{2}$

Question26

Show that the semi-vertical angle of the right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$

Solution:

Consider x be the radius and y be the height of the cone and semi-vertical angle be And, Total Surface area of cone (S) = $\pi x\sqrt{x^2+y^2}+\pi x^2$

$x\sqrt{x^2+y^2}+x^2=\frac{S}{\pi}=k$ (Say)

$x\sqrt{x^2+y^2}=k-x^2$

$x^2(x^2+y^2)=(k-x^2)^2$

$x^4+x^2y^2=k^2+x^4-2kx^2$

$x^2y^2=k^2-2kx^2$

$x^2=\frac{k^2}{y^2+2k}$ ………….. (1)

Volume of cone (V) = $\frac{1}{3}\pi x^2 y$

$$\frac{1}{3}\pi\left(\frac{k^2}{y^2+2k}\right)$$

$$=\frac{1}{3}\pi k^2\left(\frac{y}{y^2+2k}\right)$$

$$\frac{dV}{dy}=\frac{1}{3}\pi k^2\frac{d}{dy}\frac{y}{y^2+2k}$$

$=\frac{1}{3}\pi k^2\left[\frac{(y^2+2k)1-y.2y}{(y^2+2k)^2}\right]$[Using quotient rule]

$\frac{dV}{dy}=\frac{1}{3}\pi k^2\frac{(2k-y^2)}{(y^2+2k)^2}$ ……………. (2)

Now $\frac{dV}{dy}=0$

$$\frac{1}{3}\pi k^2\frac{(2k-y^2)}{(y^2+2k)^2}=0$$

$2k-y^2=0$

$y^2=2k$

$y=\pm\sqrt{2k}$

$y=\sqrt{2k}$ [Height can't be negative]

Here $y=\sqrt{2k}$ is the turning point.

As,$\frac{dV}{dy}>0$, therefore, Volume is maximum at $y=\sqrt{2k}$

Form equation (1), $x^2=\frac{k^2}{2k+2k}=\frac{k^2}{4k}=\frac{k}{4}$

$$x=\frac{\sqrt{k}}{2}$$

Now Semi- vertical angle of the cone $\sin\theta=\frac{x}{\sqrt{x^2+y^2}}$

$$=\frac{\frac{\sqrt{k}}{2}}{\sqrt{\frac{k}{4}+2k}}=\frac{\sqrt{k}}{2}\times\sqrt{\frac{4}{9k}}=\frac{1}{3}$$

Which implies

$$\theta=\sin^{-1}\frac{1}{3}$$

Choose the correct answer in the Exercise 27 to 29.

Question27

The point on the curve $x^2 = 2y$ which is nearest to the point (0, 5)
$(2\sqrt{2}, 2)$ (b) $(2\sqrt{2}, 0)$ (c) $(0, 0)$ (d) $(2, 2)$

Solution:

Option (A) is correct.
Explanation:
Equation of the curve is $x^2 = 2y$ ………. (1)
Consider P(x, y) be any point on the curve (1), then according to question,
Distance between given point (0, 5) and P $=\sqrt{(x-2)^2+(y-5)^2} = z$ (say)
$= z^2 = x^2 + (y-5)^2$
$= 2y + (y-5)^2$ [From equation (1)]
$= z^2 = 2y + y^2 + 25 - 10y$
$= z^2 = y^2 - 8y + 25 = z$ (Say)
$= \frac{dZ}{dy} = 2y - 8$ and $\frac{d^2Z}{dy^2} = 2$
Now $\frac{dZ}{dy} = 0$
$= 2y - 8 = 0$
$= y = 4$
At $y = 4$
$\frac{d^2Z}{dy^2} = 2$ [Positive]
$\therefore$ Z Is minimum and z is minimum at $y = 4$
From equation (1), we have $x^2 = 8$
$= x = \pm 2\sqrt{2}$
$(2\sqrt{2}, 4)$And $(-2\sqrt{2}, 4)$ are two points on curve (1) which are nearest to (0, 5).

Question28

For all real values of x, the minimum value of $\frac{1-x+x^2}{1+x+x^2}$ is:
(A) 0 (B) 1 (C) 3 (D) 1/3

Solution:

Option (D) is correct.
Explanation:
Given function is:
$$f(x) = \frac{1-x+x^2}{1+x+x^2}$$

$$\Rightarrow f'(x) = \frac{(1+x+x^2)\frac{d}{dx}(1-x+x^2) - (1-x+x^2)\frac{d}{dx}(1+x+x^2)}{(1+x+x^2)^2}$$
$$\Rightarrow f'(x) = \frac{(1+x+x^2)(-1+2x) - (1-x+x^2)(1+2x)}{(1+x+x^2)^2}$$
$$\Rightarrow f'(x) = \frac{-1+2x-x+2x^2-x^2+2x^3-1-2x+x+2x^2-x^2-2x^3}{(1+x+x^2)^2}$$
$$\Rightarrow f'(x) = \frac{-2+2x^2}{(1+x+x^2)^2} = -\frac{-2(1-x^2)}{(1+x+x^2)^2}$$
Now$f'(x) = 0$
$$\Rightarrow \frac{-2(1-x^2)}{(1+x+x^2)^2} = 0$$
$\Rightarrow -2(1-x^2) = 0$
$\Rightarrow 1 - x^2 = 0$
$\Rightarrow x^2 = 1$
$\Rightarrow x = \pm 1$
$\therefore x = 1$ and$x = -1$ [Turning points]
At $x = -1$
From equation (1),
$$f(-1) = \frac{1+1+1}{1-1+1} = 3$$
At $x = 1$,
From equation (1)
$f(1) = \frac{1-1+1}{1+1+1} = \frac{1}{3}$ [Minimum value]

Question29

The maximum value of $[x(x-1)+x]^{1/3}, 0 \le x \le 1$

(a) $\left(\frac{1}{3}\right)^{1/3}$ (b) ½ (c) 1 (d) 1/3

Solution:

Option (C) is correct.
Explanation:
Consider $f(x) = [x(x-1)+1]^{\frac{1}{3}}$
$= (x^2 - x + 1)^{\frac{1}{3}}, 0 \le x \le 1$ ………… (i)
$$\therefore f'(x) = \frac{1}{3}(x^2-x+1)^{\frac{-2}{3}}\frac{d}{dx}(x^2-x+1)$$
$$= \frac{(2x-1)}{3(x^2-x+1)^{\frac{2}{3}}}$$

Now $f'(x) = 0$

$= \frac{(2x-1)}{3(x^2-x+1)^{\frac{2}{3}}} = 0$

$2x - 1 = 0$

Here $x = \frac{1}{2}$ is a turning point and it belongs to the given enclosed interval $0 \leq x \leq 1$ that is, [0, 1].

At, $x = \frac{1}{2}$ from equation (i),

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{4} - \frac{1}{2} + 1\right)^{\frac{1}{3}} = \left(\frac{1-2+4}{4}\right)^{\frac{1}{3}} = \left(\frac{3}{4}\right)^{\frac{1}{3}} < 1$$

At $x = 0$, from equation (i),

$$f(0) = (1)^{\frac{1}{3}} = 1$$

At $x = 1$, from equation (i),

$f(1) = (1-1+1)^{\frac{1}{3}} = (1)^{\frac{1}{3}} = 1$

$\therefore$ Maximum value of $f(x)$ is 1

Miscellaneous Exercise

Quetion1

Using differentials, find the approximate value of each of the following:

i. $\left(\frac{17}{81}\right)^{1/4}$

ii. $(33)^{-1/5}$

Solution:

(i) $\left(\frac{17}{81}\right)^{\frac{1}{4}}$

Consider $y = x^{\frac{1}{4}}$ ………………... (1)

$$\frac{dy}{dx} = \frac{1}{4x^{\frac{3}{4}}}$$

$$dy = \frac{dx}{4\left(x^{\frac{1}{4}}\right)^3}$$

$= \frac{\Delta x}{4\left(x^{\frac{1}{4}}\right)^3}$ ……… (2)

Changing x to $x + \Delta x$ and y to $y + \Delta y$ in equation (1), we have

$y + \Delta y = (x + \Delta x)^{\frac{1}{4}} = \left(\frac{17}{81}\right)^{\frac{1}{4}} = \left(\frac{16}{81} + \frac{1}{81}\right)^{\frac{1}{4}}$ …………… (3)

Here $x = \frac{16}{81}$ and $\Delta x = \frac{1}{81}$

So, $x^{\frac{1}{4}} = \left(\frac{16}{81}\right)^{\frac{1}{4}} = \frac{2}{3}$

From equation (3),

$$\left(\frac{17}{81}\right)^{\frac{1}{4}} = y + \Delta y \sim y + dy \sim x^{\frac{1}{4}} + \frac{\Delta x}{4\left(x^{\frac{1}{4}}\right)^3}$$

$$\left(\frac{17}{81}\right)^{\frac{1}{4}} \sim \frac{2}{3} + \frac{\frac{1}{81}}{4\left(\frac{2}{3}\right)^3}$$

$$\sim \frac{2}{3} + \frac{1}{81} \times \frac{27}{32}$$

$$\sim \frac{2}{3} + \frac{1}{6} = \frac{27}{32} \quad 0.677$$

(ii) $(33)^{\frac{-1}{5}}$

Consider $y = x^{\frac{-1}{5}}$ ………… (1)

$$\frac{dy}{dx} = \frac{-1}{5x^{\frac{6}{5}}}$$

$$dy = \frac{-dx}{5\left(x^{\frac{1}{5}}\right)^6}$$

$$= \frac{-\Delta x}{5\left(x^{\frac{1}{5}}\right)^6} \quad \text{…………} \quad (2)$$

Changing x to $x + \Delta x$ and y to $y + \Delta$ in equation (1), we have

$y + \Delta y = (x + \Delta x)^{\frac{-1}{5}} = (33)^{\frac{-1}{5}} = (32 + 1)^{\frac{-1}{5}}$ ………… (3)

Here $x = 32$ and $\Delta x = 1$

$x^{\frac{-1}{5}} = (32)^{\frac{-1}{5}} = \frac{1}{2}$

From equation (3),

$$(33)^{\frac{-1}{5}} = y + \Delta y \sim y + dy \sim x^{\frac{-1}{5}} + \frac{\Delta x}{5\left(x^{\frac{1}{3}}\right)^6}$$

$$= (33)^{\frac{-1}{5}} \sim \frac{1}{2} - \frac{1}{5(2)^5}$$

$$\sim \frac{1}{2} - \frac{1}{5} \times \frac{1}{64} \sim \frac{1}{2} - \frac{1}{320} = \frac{159}{320} = \quad 0.497$$

Question2

Show that the function given by $f(x) = \frac{\log x}{x}$ has maximum value at x = e.

Solution:

Here $f(x) = \frac{\log x}{x}, x > 0$ ………… (1)

$f'(x) = \frac{x.\frac{1}{x} - \log x, 1}{x^2} = \frac{1 - \log x}{x^2}$ ……………….. (2)

And $f"(x) = \frac{x^2\left(\frac{-1}{x}\right) - (1 - \log x).2x}{x^4} = \frac{-x - 2x + 2x \log x}{x^4}$

$= \frac{2x \log x - 3x}{x^4}$

Which implies, $f'(x) = \frac{x(2 \log x - 3}{x^4} = \frac{2 \log x - 3}{x^4}$ ………… (3)

Now $f'(x) = 0$

$\frac{1 - \log x}{x^2} = 0$

$1 - \log x = 0$

$\log x = 1$

$\log x = \log e$

$x = e$

From equation (3),

$f"(x) = \frac{2 \log e - 3}{e^3} = \frac{2 - 3}{e^3} = \frac{-1}{e^3} < 0$

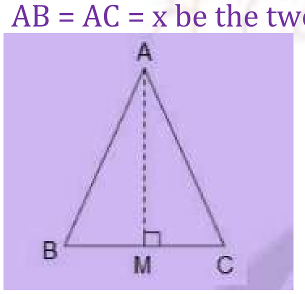
Point of local maxima and maximum value of f(x) at x = e.

Question3

The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base?

Solution:

: Consider BC = b be the fixed base and
AB = AC = x be the two sides of isosceles triangle.



Since $\frac{dx}{dt} = -3\ cm/s$ ………… (1)

Area of triangle = 1/2 x BC x AM

$= \frac{b}{2}\sqrt{x^2 - \frac{b^2}{4}}$

$$= \frac{b}{2}\sqrt{\frac{4x^2 - b^2}{4}} = \frac{b}{4}\sqrt{4x^2 - b^2}$$

$$\frac{d\Delta}{dt} = \frac{d}{dt}\left(\frac{b}{4}\sqrt{4x^2 - b^2}\right)$$

$= \frac{b}{4} \times \frac{d}{dx}\left(\sqrt{4x^2 - b^2}\right) \times \frac{dx}{dt}$[By chain rule]

$$\frac{d\Delta}{dt} = \frac{b}{4} \times \frac{1}{2\sqrt{4x^2 - b^2}} \times 8x \times (-3)$$

$= \frac{-3bx}{\sqrt{4x^2-b^2}}$cm2/s

Now, when x = b

$\frac{d\Delta}{dt} = \frac{-3b.b}{\sqrt{4b^2-b^2}} = \frac{-3b^2}{\sqrt{3b}} = -\sqrt{3b}\ \ cm^2/s$

Therefore, the area is decreasing at the rate of $\sqrt{3b}$ cm2 /s.

Question4

Find the equation of the normal to the curve x 2 = 4y at the point (1, 2).

Solution:

Equation of the curve is x 2 = 4y .... (1)

Differentiate w.r.t. x,

2x = 4 dy/dx

$= \frac{dy}{dx} = \frac{x}{2}$ =m (say)

Slope of the normal to the curve at (1, 2) is -1/m = -2/x

Equation of the normal to the curve (1) at (1, 2) is

x + y = 3

Question5

Show that the normal at any point θ to the curve

$x = a\cos\theta + a\theta\sin\theta$, $y = a\sin\theta - a\theta\cos\theta$is at a constant distance from the origin.

Solution:

The parametric equations of the curve are

$x = a\cos\theta + a\theta\sin\theta\,, y = a\sin\theta - a\theta\cos\theta$

$x = a(\cos\theta + \theta\sin\theta)\,, y = a(\sin\theta - \theta\cos\theta)$

$$\frac{dx}{d\theta} = a\,[-\sin\theta + \theta\cos\theta + \sin\theta] = a\theta\cos\theta$$

And $\frac{dy}{d\theta} = a[\cos\theta - (-\theta\sin\theta + \cos\theta)] = a[\cos\theta + \theta\sin\theta - \cos\theta] = a\theta\sin\theta$

Slope of tangent at point (x, y)

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\theta \sin\theta}{a\theta \cos\theta} = \tan\theta$

Slope of normal at any point θ

$= \frac{-1}{\tan\theta} = -\cot\theta = -\frac{\cos\theta}{\sin\theta}$

And Equation of normal at any point θ, that is, at (x, y)

$= [a(\cos\theta + \theta\sin\theta), a(\sin\theta - \theta\cos\theta]$

is $y - a\,(\sin\theta + \theta\cos\theta) = \frac{-\cos\theta}{\sin\theta}[x - a)\cos\theta + \theta\sin\theta]$

$y\sin\theta - a\sin^2\theta + \alpha\theta\cos\theta\sin\theta = -x\cos\theta + a\cos^2\theta + a\theta\sin\theta\cos\theta$

$x\cos\theta + y\sin\theta = a(\sin^2\theta + \cos^2\theta)$

$x\cos\theta + y\sin\theta = a$

$x\cos\theta + y\sin\theta - a = 0$

Distance of normal from origin (0, 0)

$\frac{|0+0-a|}{\sqrt{\cos^2\theta+\sin^2}}$ Which is a constant.

Question6

Find the intervals in which the function f given by $f(x) = \frac{4\sin x - 2x - x\cos x}{2+\cos x}$ Is

(i) Increasing (ii) decreasing.

Solution:

Given function is:

$f(x) = \frac{4\sin x - 2x - x\cos x}{2 + \cos x}$

$= \frac{4\sin x - x(2 + \cos x)}{2 + \cos x}$

$= \frac{4\sin x}{2 + \cos x} - \frac{x(2 + \cos x)}{2 + \cos x}$

$= \frac{4\sin x}{2 + \cos x} - x$

On differentiating:

$f'(x) = \frac{(2 + \cos x)\frac{d}{dx}(4\sin x) - 4\sin x\frac{d}{dx}(2 + \cos x)}{(2 + \cos x)^2} - 1$

$f'(x) = \frac{(2 + \cos x)(4\cos x) - 4\sin x(-\sin x)}{(2 + \cos x)^2} - 1$

$= \frac{8\cos x + 4\cos^2 x + 4\sin^2 x}{(2 + \cos x)^2} - 1$

Which implies,

$f'(x) = \frac{8\cos x + 4}{(2 + \cos x)^2} - 1$

$= \frac{8\cos x + 4 - (2+\cos x)^2}{(2+\cos x)^2}$

$\frac{8\cos x + 4 - -4 - \cos^2 x - 4\cos x}{(2+\cos x)^2}$

$= f'(x) = \frac{4\cos x - \cos^2 x}{(2+\cos x)^2}$

$= \cos x \frac{(4-\cos x)}{(2+\cos x)^2}$ ………… (i)

Now $4 - \cos x > 0$ *for all real x as* $-1 \le \cos x \le 1$. Also $(2+\cos x)^2 > 0$

(i) f(x) is increasing if $f'(x) \ge 0$, that is, from equation (1) $\cos x \ge 0$

X lies in I and IV quadrants, that is, f(x) is increasing for $0 \le x \le \frac{\pi}{2}$

and $\frac{3\pi}{2} \le x \le 2\pi$

and (2) f(x) is decreasing if $f'(x) \le 0$, that is, from equaon(1), $\cos x \le 0$

$\Rightarrow$ x lies in II and III quadrants, that is, f(x) is decreasing for $\frac{\pi}{2} \le x \le \frac{3\pi}{2}$

Question7

Find the intervals in which the function f given by $f(x) = x^3 + \frac{1}{x^3}, x \ne 0$ is

(i) Increasing (ii) decreasing.

Solution:

(i), $f(x) = x^3 + \frac{1}{x^3}, x \ne 0$

$f(x) = x^3 + x^{-3}$

$f'(x) = 3x^2 - 3x^{-4} = 3\left(x^2 - \frac{1}{x^4}\right)$

$3\left(\frac{x^6 - 1}{x^4}\right) = \frac{3}{x^4}[(x^2)^3 - 1^3]$

$= f'(x) = \frac{3}{x^4}(x^2 - 1)(x^4 + x^2 + 1)$

$= \frac{3}{x^4}(x^4 + x^2 + 1)(x+1)(x-1)$ ………… (1)

Now $f(x) = 0$

$= \frac{3}{x^4}(x^4 + x^2 + 1)(x+1)(x-1) = 0$

$= 3(x^4 + x^2 + 1)(x+1)(x-1) = 0$

Here, $3(x^4 + x^2 + 1)$ is positive for all real $x \ne 0$

$x + 1 = 0$ or $x - 1 = 0$ [Turning points]

Therefore, $x = -1$ or $x = 1$ divide the real line into three subs intervals $(-\infty, -1), (-1, 1)$ and

for $(-\infty, -1)$, from equation (1) at $x = -2$ (say)

$f'(x) = (+)(-)(-) = (+)$

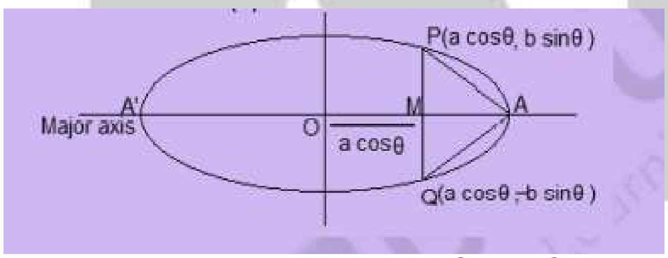
Therefore, f (x) is increasing at x = -1
For (-1, 1) from equation (1)x $= \frac{1}{2}$ (say)
F' (x) =(+)(+)(-)=(-)
Therefore,f(x)isdecresingat $x = -11$
For$(1, \infty)$, from equation (1) at $x = 2$ (say)
$f'(x) = (+)(+)(-) = (+)$
Therefore, $f(x)$ is increasing at $x = 1$
Therefore, $f(x)$ is (1) an increasing function for$x \leq -1$ and for$x \geq 1$ and (2) decreasing function for$-1 \leq x \leq 1$

Question8

Find the maximum area of an isosceles triangle inscribed in the $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ellipse with its vertex at one end of the major axis.

Solution

Equation of the ellipse is
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ …………….. (1)



Comparing equation (1) with $\cos^2 + \sin^2 = 1$ we have
$\frac{x}{a} = \cos\theta$ and $\frac{y}{b} = \sin\theta$
or $x = a\cos\theta$ and $y = \sin\theta$
Any point on ellipse is P (a cos θ , b sin θ)
Draw PM perpendicular to x-axis and produce it to meet the ellipse in the point Q.
OM = a cos θ and PM= b sin θ
We know that the ellipse (1) is symmetrical about x-axis, therefore,
PM = QM and So triangle APQ is isosceles.
Area of APQ (z) = 1/2 x Base x Height
$= \frac{1}{2}$ PQ. AM $= \frac{1}{2}$
2PM.AM = PM (OA- OM)
$\Rightarrow z = b\sin\theta\,(a - a\cos\theta)$
$= ab(\sin\theta \sin\theta \cos\theta)$
$\Rightarrow z = \frac{ab}{2}(2\sin\theta - 2\sin\theta\cos\theta)$

$= \frac{ab}{2}(2\sin\theta - \sin 2\theta)$
$\Rightarrow \frac{dz}{d\theta} = \frac{ab}{2}(2\cos\theta - 2\cos 2\theta)$
$= ab(\cos\theta - \sin 2\theta)$
$\Rightarrow \frac{d^2z}{d\theta^2} = ab(-\sin\theta + 2\sin 2\theta)$
Now $\frac{dz}{d\theta} = 0$
$\Rightarrow\ ab(\cos\theta - \cos 2\theta) = 0$
$\Rightarrow \cos\theta - \cos 2\theta = 0$
$\Rightarrow\ \cos\theta = \cos 2\theta$
$\Rightarrow \cos\theta = \cos(360^0 - 2\theta)$
$\Rightarrow \theta = 2\theta$ or $\theta = 360^0 - 2\theta$
that is, $\theta = 0$ or $3\theta = 360^0$
$\Rightarrow \theta = 120^0$
$\theta = 0$ is impossible
$\therefore \theta = 120^0$
At $\theta = 120^0, \frac{d^2z}{d\theta^2} = ab(-\sin 120^0 + 2\sin 240^0)$
$= ab\left(\frac{\sqrt{3}}{2} - \frac{2\sqrt{3}}{2}\right) = ab\left(\frac{-3\sqrt{3}}{2}\right)$
$\therefore$ z is maximum at $\theta = 120^0$
$\therefore$ From equation (1), Maximum area
$= \frac{ab}{2}(2\sin 120^0 - \sin 240^0)$
$= \frac{ab}{2}\left(\frac{2\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right)$
$= \frac{ab}{2}\left(\frac{3\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right)$
$= \frac{ab}{2}\left(\frac{3\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{4}ab$

Question9

A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m3. If building of tank costs Rs. 70 per sq. meter for the base and Rs. 45 per square meter for sides. What is the cost of least expensive tank?

Solution:

Depth of tank = 2 m
Consider x m be the length and y m be the breadth of the base of the tank.

Volume of tank = 8 cubic meter
$x.y.2 = 8$
$xy = 4$
$y = \frac{4}{x}$
Cost of building the
Base of the tank at the rate of Rs. 70 per sq. meter is 70 xy.
And cost of building the four walls of the tank at the rate of Rs. 45 per sq. meter is
$45(x2 + x2 + y2 + y2)$
$= (180x + 180y)$
Consider z be the total cost of building the tank
$z = 70xy + 180x + 180y$
$\therefore \frac{dz}{dx} = 0 + 180 - \frac{720}{x^2}$ and $\frac{d^2z}{dx^2} = \frac{1440}{x^3}$
Now$\frac{dz}{dx} = 0$
$\Rightarrow 180 - \frac{720}{x^2} = 0$
$\Rightarrow \frac{720}{x^2} = 180$
$\Rightarrow 180x^2 = 720$
$\Rightarrow x^2 = 4$
$\Rightarrow x = 2$ [Length cannot be negative]
At $x = 2$
$\frac{d^2z}{dx^2} = \frac{1440}{8} = 180$ [Positive]
$\therefore z$ is minimum at $x = 2$
Minimum cost
$= 280 + 180 \times 2 + \frac{720}{2}$
$= 280 + 360 + 360 = \text{Rs. } 1000$

Question10

The sum of the perimeter of a circle and square is k, where k is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.

Solution:

$= 2\pi + 4y = k$
$= 4y = k - 2\pi x$
$= y = \frac{k-2\pi x}{4}$ .... (1)
Consider z be the sum of areas of circle and square.
$\therefore z = \pi x^2 + y^2$
$\Rightarrow z = \pi x^2 + \frac{(k-2\pi x)^2}{16}$ [From equation]

$\Rightarrow z = \dfrac{16\pi x^2 + k^2 + 4\pi^2 x^2 - 4k\pi x}{16}$

$= \dfrac{1}{16}[(16\pi + 4\pi^2)x^2 - 4\pi x + k^2]$

$\frac{dz}{dx} = \frac{1}{16}[(16\pi + 4\pi^2)2x - 4k\pi]$ and

$\dfrac{d^2z}{dx^2} = \dfrac{1}{16}(16\pi + 4\pi^2)2$

Now $\frac{dz}{dx} = 0$

$\Rightarrow \dfrac{1}{16}[(16\pi + 4\pi^2)2x - 4k\pi] = 0$

$\Rightarrow (16\pi + 4\pi^2)2x - 4k\pi = 0$

$\Rightarrow 4\pi(4+\pi)2x = 4k\pi$

$\Rightarrow x = \dfrac{4k\pi}{4\pi(4+\pi)2} = \dfrac{k}{2(4+\pi)}$

At $x = \frac{k}{2(4+\pi)} \frac{d^2z}{dx^2} = \frac{1}{16}(16\pi + 4\pi^2)2$ [Positive]

$\therefore$ z is minimum when $x = \frac{k}{2(4+\pi)}$

$\therefore$ From equation (1),

$y = \dfrac{1}{4}\left[k - 2\pi\dfrac{k}{2(4+\pi)}\right]$

$= \dfrac{1}{4}\left[k - \dfrac{\pi k}{4+\pi}\right]$

$= \dfrac{1}{4}\left[\dfrac{k(4+\pi) - \pi k}{(4+\pi)}\right]$

$= y = \dfrac{4k + \pi k - \pi k}{4(4+\pi)}$

$= \dfrac{4k}{4(4+\pi)}$

$= 2\dfrac{k}{2(4+\pi)} = 2x$

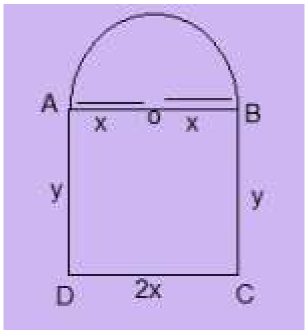
Therefore, sum of areas is minimum when side of the square is double the radius of the circle.

Question11

A window is in the form of a rectangle surmounted by a semi-circular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

Solution:

Consider x m be the radius of the semi-circular opening. Then one side of rectangle part of window is 2x and y m is the other side



Perimeter of window = Semi-circular arc AB + Length (AD + DC + BC)

$\Rightarrow \frac{1}{2}(2\pi x) + y + 2x + y = 10$

$\Rightarrow \pi x + 2x + 2y = 10$

$\Rightarrow 2y = 10 - \pi x - 2x$

$\Rightarrow y = \frac{10-(\pi+2)x}{2}$ ………….. (1)

Area of window (z) = Area of semi-circle + Area of rectangle

$= \frac{1}{2}(\pi x^2) + 2xy$

$= z = \frac{1}{2}\pi x^2 + 2x\left[\frac{10-(\pi+2)x}{2}\right]$

$\Rightarrow \frac{1}{2}[\pi x^2 + 20x - 2(\pi+2)x^2]$

$\Rightarrow z = \frac{1}{2}[\pi x^2 + 20x - 2\pi x^2 - 4x^2]$

$= \frac{1}{2}[-\pi x^2 - 4x^2 + 20x]$

$\frac{dz}{dx} = \frac{1}{2}[-2\pi x - 8x + 20]$ and

$\frac{d^2z}{dx^2} = \frac{1}{2}(-2\pi - 8) = -(x + 4)$

Now $\frac{dz}{dx} = 0$

$\Rightarrow \frac{1}{2}[-2\pi - 8x + 20] = 0$

$\Rightarrow -2\pi x - 8x + 20 = 0$

$\Rightarrow -2x(\pi + 4) = -20\ \ 0$

$\Rightarrow x = \frac{20}{2(\pi+4)}$

$\Rightarrow\ \ x = \frac{10}{\pi+4}$

At $x = \frac{10}{\pi+4} \frac{d^2z}{dx^2} = -(\pi + 4)$ [Negative]

$\therefore z$ is maximum a$x = \frac{10}{\pi+4}$

From equation (1),

$$y = \frac{10-(\pi+2)\frac{10}{\pi+4}}{2}$$
$$= \frac{10(\pi+4)-10(\pi+2)}{2(\pi+4)}$$
$$\Rightarrow y = \frac{10\pi+10-10\pi-20}{2(\pi+4)}$$
$$= \frac{20}{2(\pi+4)} = \frac{10}{\pi+4}\ \text{m}$$

Therefore, Length of rectangle = $2x = \frac{20}{\pi+4}$ m and Width of rectangle =$y = \frac{10}{\pi+4}$ m

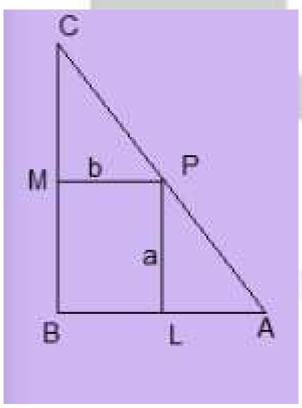
And Radius of semi-circle = $x = \frac{10}{\pi+4}$ m

Question12

A point on the hypotenuse of a triangle is at distances a and b from the sides of the triangle. Show that the maximum length of the hypotenuse is $\left(a^{2/3}+b^{2/3}\right)^{3/2}$

Solution:

Consider a right triangle ABC.
P be a point on the hypotenuse AC such that PL ⊥ AB =a and PM ⊥BC = b and
Consider ∠ BAC = ∠MPC =θ, then in right angled ΔALP, $\frac{AP}{PL} = \cos ec\theta$



From triangle, AP
PL $\cos e^{\theta} = a \cos e^{\theta}$
And in right angledΔ PMC $\frac{PC}{Pm} = \sin\theta$
⇒PM=PM $\sec\theta = b\sec\theta$
Consider AC = z, then
Z = AP+PC = $\cos ec\theta + b\sec\theta, 0 < \theta < \frac{\pi}{2}$ .... (1)
$$\frac{dz}{d\theta} = -a\cos ec\theta\cot\theta + b\sec\theta\tan\theta$$

Now $\frac{dz}{d\theta} = 0$

$\Rightarrow \; -a\cos ec\theta \cot\theta + b\sec\theta\tan\theta = 0$

$\Rightarrow \frac{b\sin\theta}{\cos^2\theta} = \frac{a\cos\theta}{\sin^2\theta}$

$\Rightarrow b\sin^3\theta = a\cos^3\theta \Rightarrow \frac{a}{b} = \frac{\sin^3\theta}{\cos^3\theta}$

$\Rightarrow \frac{a}{b} = \tan^3\theta$

$\Rightarrow \; \tan\theta = \left(\frac{a}{b}\right)^{\frac{1}{3}}$ ……………. (2)

$\frac{d^2z}{d\theta^2} a[\cos ec\theta\,(-\cos ec^2) + \cot\theta\,(-\cos ec\theta\cot\theta)] + b[\sec\theta\sec^2\theta + \tan\theta\sec\theta\tan\theta]$

$\Rightarrow \frac{d^2z}{d\theta^2} = a(\cos ec^3 + \cos ec\theta\cot^2\theta) + b(\sec^3\theta + \sec\theta\tan^2\theta)$

$\Rightarrow \frac{d^2z}{d\theta^2} > 0[a > 0, b > 0 \text{ and } \theta \text{ is } +ve \text{ as } 0 < \theta < \frac{\pi}{2})$

Z is minimum

When $\tan\theta = \left(\frac{a}{b}\right)^{\frac{1}{3}}$

$\sec^2\theta = 1 + \tan^2 = 1 + \left(\frac{a}{b}\right)^{\frac{2}{3}} = \frac{b^{\frac{2}{3}} + a^{\frac{2}{3}}}{b^{\frac{2}{3}}}$

$\Rightarrow \; \sec\theta = \frac{\left(b^{\frac{2}{3}} + a^{\frac{2}{3}}\right)^{\frac{1}{2}}}{b^{\frac{1}{3}}}$

Also $\Rightarrow \cos ec^2 = 1 + \cot^2 = 1 + \left(\frac{b}{a}\right)^{\frac{2}{3}}$

$= \frac{a^{\frac{2}{3}} + b^{\frac{2}{3}}}{a^{\frac{2}{3}}}$

$\Rightarrow \cos ec\theta = \frac{\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{1}{2}}}{a^{\frac{1}{3}}}$

Putting these values in equation (1),

Minimum length of hypotenuse = $a\frac{\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{1}{2}}}{a^{\frac{1}{3}}} + b\frac{\left(b^{\frac{2}{3}}+a^{\frac{2}{3}}\right)^{\frac{1}{2}}}{b^{\frac{1}{3}}}$

$= \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{1}{2}}\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)$

$= \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$

Question13

Find the points at which the function f given by $f(x) = (x-2)^4(x+1)^3$ has:

i. local maxima
ii. local minima
iii. point of inflexion

Solution:

$f(x) = (x-2)^4(x+1)^3$

$\therefore f'(x) = (x-2)^4 \dfrac{d}{dx}(x+1)^3 + \dfrac{d}{dx}(x-2)^4(x+1)^3$

$= (x-2)^4(x+1)^2 + 4(x-2)^3(x+1)^3$

$= (x-2)^4(x+1)^2[3(-2) + 4(x+1)]$

$= (x-2)^3(x+1)^2(7x-2)$

Now $f(x) = 0$

$\Rightarrow (x-2)^3(x+1)^2(7x-2) = 0$

$\Rightarrow x - 2 = 0$ or $x + 1 = 0$ or $7x - 2 = 0$

$\Rightarrow x = 2$ or $x = -1$ or $x = \dfrac{2}{7}$

Now, for values x close to $\frac{2}{7}$ and to the left of $\frac{2}{7}$, $f'(x) > 0$

Also for values of x to $\frac{2}{7}$ and to the right of $\frac{2}{7}$, $f'(x) > 0$

Therefore, $x = \frac{2}{7}$ is the point of local maxima.

Now for values

Of x close to 2 and to the left of 2, $f'(x) > 0$. Also for values of x close to 2 and to the right of 2, $f'(x) > 0$

Therefore, x = 2 is the point of local minima.

Now as the values of x varies through -1, f'(x) does not change its sign. Therefore, x = -1 is the point of inflexion

Question14

Find the absolute maximum and minimum values of the function f given by $f(x) = \cos^2 x + \sin x, x \in [0, \pi]$

Solution:

$f(x) = \cos^2 x + \sin x, x \in [0, \pi]$ ……….. (1)

$f'(x) 2\cos x \dfrac{d}{dx} \cos x + \cos x$

$= -2\cos x \sin x + \cos x = \cos x(-2\sin x + 1)$

Now $f'(x) = 0$

$\Rightarrow \cos x(-2\sin x + 1) = 0$

$\cos x = 0$ or $-2\sin x + 1 = 0$

$\Rightarrow x = \frac{\pi}{2}$ or $2\sin x = 1$

$\Rightarrow \sin x = \frac{1}{2}$

Here $x = \frac{\pi}{6}$ is a turning point

Now $f\left(\frac{\pi}{2}\right) = \cos^2\frac{\pi}{2} + \sin\frac{\pi}{2}$

$= 0 + 1 = 1$

$f\left(\frac{\pi}{6}\right) = \cos^2\frac{\pi}{6} + \sin\frac{\pi}{6} = \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2} = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$

$f(0) = \cos^2 0 + \sin 0 = 1 + 0 = 1$

$f(\pi) = \cos^2\pi + \sin\pi = (-1)^2 + 0 = 1$

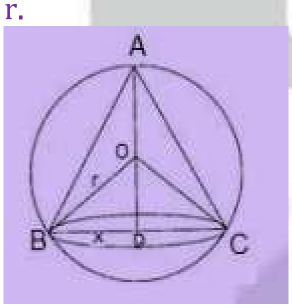
Therefore, absolute maximum is 5/4 and absolute minimum is 1.

Question15

Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4\pi}{3}$

Solution:

Consider x be the radius of base of cone and y be the height of the cone inscribed in a sphere of radius r.



OD = AD – AO = y – r

In right angled triangle OBD,

OD2 + BD2 = OB2

$\Rightarrow (y - r)^2 + x^2 = r^2$

$\Rightarrow y^2 + r^2 - 2ry + x^2 = r^2$

$\Rightarrow x^2 = 2ry - y^2$ …………. (1)

Volume of cone (V) = $\frac{1}{3}\pi x^2 y = \frac{1}{3}\pi(2ry - y^2)y$ [From equation (1)]

$\Rightarrow V = \frac{\pi}{3}(2ry^2 - y^3)$

$\Rightarrow \frac{dV}{dy} = \frac{\pi}{3}(4ry - 3y^2)$ and $\frac{d^2V}{dy^2} = \frac{\pi}{3}(4r - 6y)$

$\text{Now}\dfrac{dV}{dy} = 0$

$\Rightarrow \dfrac{\pi}{3}(4ry - 3y^2) = 0$

$\Rightarrow \dfrac{\pi y}{3}(4r - 3y) = 0$

$\Rightarrow 4r - 3y = 0$

$\Rightarrow y = \dfrac{4r}{3}$

$\text{At } y = \dfrac{4r}{3}\dfrac{d^2V}{dy^2} = \dfrac{\pi}{3}(4r - 8r)$

$= \frac{-4\pi r}{3}$ [Negative]

Volume is maximum at $y = \frac{4r}{3}$

Question16

Consider f be a function defined on $[a, b]$ such that $f'(x) > 0$ for all $x \in (a, b)$Then prove that f is an increasing function on (a, b).

Solution:

Consider I be the interval (a, b)
Given; $f^{'}(x) > 0$ for all x in an interval I. Consider $x_{1,}x_2 \in I$ with $x_1 < x_2$
By Lagrange's Mean Value Theorem, we have

$\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = f^{'}(c)\text{where } x_1 < c < x_2$

$\Rightarrow f(x_2) - f(x_1) = (x_2 - x_1)f^{'}(c)\text{where } x_1 < c < x_2$

$\text{Now} x_1 < x_2$

$\Rightarrow x_2 - x_1 > 0$

Also, $f^{'}(x) > 0$ for all x in an interval

$f^{'}(c) > 0$

$\therefore$ From equation (1),$f(x_2) - f(x_1) > 0$

$\Rightarrow f'(x_1) < f(x_2)$

Thus of every pair of point $x_{1,}x_2 \in I$ with $x_1 < x_2$

$\Rightarrow f'(x_1) < f(x_2)$

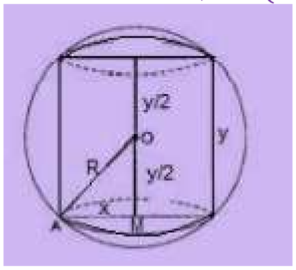
Therefore, f(x) is strictly increasing in l.

Question17

Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is$\frac{2R}{\sqrt{3}}$ Also find the maximum volume.

Solution:

Consider x be the radius and y be the height of the cylinder inscribed in a sphere having centre " O" and radius "R", for $(x > 0, y > 0)$



In right triangleOAM, $AM^2 + OM^2 = OA^2$

$\Rightarrow x^2 + \left(\frac{y}{2}\right)^2 = R^2$

$\Rightarrow x^2 = R^2 - \frac{y^2}{4}$ …………… (1)

Volume of cylinder (V) $= \pi x^2$ ……………. (2)

$\Rightarrow V = \pi\left(R^2 - \frac{y^2}{4}\right) y$

$= \pi\left(R^2 y - \frac{y^3}{4}\right)$

$\therefore \frac{dV}{dy} = \pi\left(R^2 - \frac{3y^2}{4}\right)$ and $\frac{d^2V}{dy^2} = \pi\left(-\frac{3y}{2}\right) = -\frac{3\pi y}{2}$

Now $\frac{dV}{dy} = 0$

$\Rightarrow \pi\left(R^2 - \frac{3y^2}{4}\right) = 0$

$\Rightarrow R^2 - \frac{3y^2}{4} = 0$

$\Rightarrow R^2 = \frac{3y^2}{4}$

$\Rightarrow y^2 = \frac{4R^2}{3}$

$\Rightarrow y = \frac{2R}{\sqrt{3}}$

At $y = \frac{2R}{\sqrt{3}} \frac{d^2V}{dy^2} = -\frac{3\pi}{2}\left(\frac{2R}{\sqrt{3}}\right)$

$= -\pi R\sqrt{3}$ [Negative]

V is maximum at $y = \frac{2R}{\sqrt{3}}$

From equation (3),

Maximum value of cylinder = $\pi\left[R^2 . \frac{2R}{\sqrt{3}} - \frac{1}{4}\frac{4R^2}{3}\frac{2R}{\sqrt{3}}\right]$

$= \pi R^2 \frac{2R}{\sqrt{3}}\left[1 - \frac{1}{3}\right]$

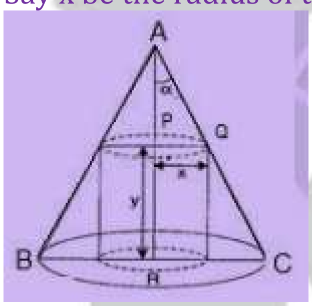
$= \frac{4\pi R^3}{3\sqrt{3}}$

Queston18

Show that the height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and having semi-vertical angle a **is one-third that of the cone and the greatest volume of the cylinder is** $\frac{4}{27}\pi h^3 \tan^2 a$

Solution:

Consider r be the radius of the right circular cone of height h.
Say x be the radius of the inscribed cylinder with height y.



In similar triangles APQ and ARC $\frac{PQ}{RC} = \frac{AP}{AR}$

$\Rightarrow \frac{x}{r} = \frac{h-y}{h}$

$\Rightarrow hx = rh - ry$

$\Rightarrow ry - rg - hx = h(r - x)$

$\Rightarrow y = \frac{h}{r}(r - x)$

Volume of cylinder (V) = $\pi x^2 y$ ……………… (2)

$\Rightarrow V = \pi x^2 \frac{h}{r}(r - x)$

$= \frac{\pi h}{r}(rx^2 - x^3)$ ……….. (3)

$\therefore \frac{dV}{dy} = \frac{\pi h}{r}(2rs - 3x^2)$and $\frac{d^2V}{dy^2} = \frac{\pi h}{r}(2r - 6x)$

Now $\frac{dV}{dy} = 0$

$\Rightarrow \frac{\pi h}{r}(2rx - 3x^2) = 0$

$\Rightarrow 2rx - 3x^2 = 0$

$\Rightarrow 2r - 3x = 0$

$\Rightarrow x = \frac{2r}{3}$

At $x = \frac{2r}{3} \frac{d^2V}{dy^2} = \frac{\pi h}{r}\left(2r - \frac{12r}{3}\right)$

$= \frac{\pi h}{r}(-2r) = -2\pi h$ [Negative]
V is maximum at $x = \frac{2r}{3}$
From equation (3),
Maximum value of
Cylinder $= \frac{\pi h}{r}\left[r.\frac{4r^2}{9} - \frac{8r^2}{27}\right]$
$= \frac{\pi h}{r} r^3 \left[\frac{4}{9} - \frac{8}{27}\right]$
$= \pi h r^2 \frac{4}{27}$
$= \frac{4}{27}\pi h(h \tan a)^2$
$= \frac{4}{27}\pi h^2 \tan^2 a \left[\therefore \frac{r}{h} = \tan a\right]$

Choose the correct answer in the Exercise 19 to 24:

Question19

A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic meters per hour. Then the depth of wheat is increasing at the rate of:
(A) 1 m/h (B) 0.1 m/h (C) 1.1 m/h (D) 0.5 m/h

Solution:

Option (A) is correct.
Explanation
Consider y be the depth of the wheat in the cylindrical tank whose radius is 10 m at time t. V = Volume of wheat in cylindrical tank at time t,
So, $t = \pi(10)^2 y = 100y$ cubic meter
We are given that $\frac{dV}{dt}$ = 314 cubic meter/hr.
So, $\frac{d}{dt}100\pi y = 314$
$100(3.14)y = 314$
Therefore, y=1 m/h

Question20

The slope of the tangent to the curve $x = t^2 + 3t - 8, y = 2t^2 - 2t - 5$ at the point (2, -1) is
(a) $\frac{22}{7}$ (b) $\frac{6}{7}$ (c) $\frac{7}{6}$ (d) $\frac{-6}{7}$

Solution:

Option (B) is correct.
Explanation:
Equations of the curves are $x = t^2 + 3t - 8$ .... (1) and $y = 2t^2 - 2t - 5$ ....(2)

$$\frac{dx}{dt} = 2t + 3 \quad \text{and}$$
$$\frac{dy}{dt} = 4t - 2$$

Slope of the tangent to the given curve at point (x, y) is

$$(x, y) = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t-2}{2t+3} \ .... \ (3)$$

At the given point $(2, -1)$, $x = 2$ and $y = -1$
At $x = 2$, from equation (1), $2 = t^2 + 3t - 8$
$t^2 + 3t - 10 = 0$
$(t + 5)(t - 2) = 0$
$t = -5, t = 2$
At $y = -1$, from equation (2), $-1 = 2t^2 - 2t - 5$
$2t^2 - 2t - 4 = 0$
$t^2 - t - 2 = 0$
$(t - 2)(t + 1) = 0$
$t = 2, t = -1$
Here, common value of t in the two sets of values is 2.
Again, from equation (3)
Slope of the tangent to the given curve at point $(2, -1) = \frac{4(2)-2}{2(2)+3} = \frac{6}{7}$

Question21

The line $y = mx + 1$ is a tangent to the curve $y^2 = 4x$ if the value of m is:
(A) 1 (B) 2 (C) 3 (D) ½

Solution:

Option (A) is correct.
Explanation:
Equation of the curve is $y^2 = 4x$ .... (1)

$$2y\frac{dy}{dx} = 4.1$$
$$\frac{dy}{dx} = \frac{2}{y}$$

Slope of the tangent to the given curve at point (x, y)

$$= \frac{dy}{dx} = \frac{2}{y}$$
$$\frac{2}{y} = m$$
$$y = \frac{2}{m} \ .... \ (2)$$

Now $y = mx + 1$

$\frac{2}{m} = mx + 1$

$mx = \frac{2}{m} - 1$

$x = \frac{2-m}{m}$ ………….. (3)

Putting the values of x and y in equation (1), $\frac{4}{m^2} = \frac{4(2-m)}{m^2}$

$2 - m = 1 \Rightarrow m = 1$

Question22

The normal at the point (1, 1) on the curve $2y + x^2 = 3$is:

(a)$x + y = 0$ (b)$x - y = 0$ (c)$x + y + 1 = 0$ (d)$x - y = 1$

Solution:

Option (B) is correct.

Explanation:

Equation of the given curve is $2y + x^2 = 3$ ……………… (1)

$2\frac{dy}{dx} + 2x = 0$

$\frac{dy}{dx} = -x$

Slope of the tangent to the given curve at point (1, 1) is

$\frac{dy}{dx} = -x = -1 = m$ (say)

Slope of the normal $= \frac{-1}{m} = \frac{-1}{-1} = 1$

Equation of the normal at (1, 1) is $y - 1 = 1(x - 1)$

$y - 1 =$

$x - y = 0$

Question23

The normal to the curve $x^2 = 4y$ passing through (1, 2) is:

(a) $x + y = 3$ (b) $x - y = 3$ (c) $x + y = 1$ (d) $x - y = 1$

Solution:

Option (A) is correct.

Explanation:

Equation of the curve is $x^2 = 4y$ … … … … … (1)

$2x = 4\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{x}{2}$$

Slope of the normal at (x, y) is

$-\frac{dx}{dy} = \frac{-2}{x}$ ………… (2)

Again slope of normal at given point (1, 2) = $\frac{y-2}{x-1}$……………. (3)

From equation (2) and (3), we have $\frac{-2}{x} = \frac{y-2}{x-1}$

$-2x + 2 = xy - 2x$

$xy = 2$

$$y = \frac{2}{x}$$

From equation (1), $x^2 = \frac{8}{x}$

$x^3 = 8$

$x = 2$

$$y = \frac{2}{x} = \frac{2}{2} = 1$$

Now, at point (2, 1), slope of the normal from equation (2) = $\frac{-2}{x} = \frac{-2}{2} = -1$

Equation of the normal is $y - 1 = -1(x - 2)$

$y - 1 = -x + 2$

or $x + y = 3$

Question24

The points on the curve where the normal to the curve make equal intercepts with axes are:

(a) $\left(4, \pm\frac{8}{3}\right)$ (b) $\left(4, -\frac{8}{3}\right)$ (c) $\left(4, \pm\frac{3}{8}\right)$ (d) $\left(\pm 4, \frac{8}{3}\right)$

Solution:

Option (A) is correct.

Explanation:

Equation of the curve is $9y^2 = x^3$……………. (1)

$$18y\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{18y} = \frac{x^2}{6y}$$

Slope of the tangent to curve (1) at any point(x, y) is

$$\frac{dy}{dx} = \frac{x^2}{6y}$$

Slope of the normal = negative reciprocal = $\frac{-6y}{x^2} = \pm 1$

As we know that, slopes of lines with equal intercepts on the axes are± 1]

So, $-6y = \pm x^2$

If, $-6y = x^2$

$y = \frac{-x^2}{6}$ …………. (2)

From equation (1) and (2), we have x = 4 and $y = \frac{8}{3}$

Required points are $\left(4, \pm\frac{8}{3}\right)$