Chapter 2
InverseTrigonometric Functions
Exercise 2.1

Question 1

Find the principal values of the following:

1. $\sin^{-1}\left(-\frac{1}{2}\right)$
2. $\sin^{-1}\left[\frac{\sqrt{3}}{2}\right]$
3. $\text{Cosec}^{-1}(2)$
4. $\tan^{-1}(-\sqrt{3})$
5. $\cos^{-1}\left(\frac{-1}{2}\right)$
6. $\tan^{-1}(-1)$
7. $\sec^{-1}\left[\frac{2}{\sqrt{3}}\right]$
8. $\cot^{-1}(\sqrt{3})$
9. $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$
10. $\text{cosec}^{-1}(-\sqrt{2})$

Solution 1:

Consider $y = \sin^{-1}\left(-\frac{1}{2}\right)$
Solve the above equation, we have
$\sin y = -1/2$
We know that $\sin \pi/6 = 1/2$
So, $\sin y = -\sin \pi/6$
$\sin y = \sin\left[-\frac{\pi}{6}\right]$
Trigonometric Functions
Since range of principle value of $\sin^{-1}$ is $\left[-\frac{\pi}{2}.\frac{\pi}{2}\right]$
Principle value of $\sin^{-1}(-\frac{1}{2})$ is $-\pi/6$.

Solution 2:

Let $y = \cos^{-1}\left[\frac{\sqrt{3}}{2}\right]$
$\cos y = \cos \pi/6$ (as $\cos \pi/6 = \sqrt{3}/2$)
$Y = \pi/6$
Since range of principle value of $\cos^{-1}$ is $[0. \pi]$
Therefore, principle value of $\cos^{-1}\left[\frac{\sqrt{3}}{2}\right]$ is $\pi/6$

Solution 3:

$Cosec^{-1}$ (2)
Let y = $Cosec^{-1}$ (2)
Cosec y = 2
We know that, cosec ∏/6 = 2
So, Cosec y = cosec ∏/6
Since range of principle value of $Cosec^{-1}$ is $\left[-\frac{\pi}{2}.\frac{\pi}{2}\right]$
Therefore, Principle value of $Cosec^{-1}$ (2) is Π/6.

Solution 4:

$\tan^{-1}(-\sqrt{3})$
Let y = $\tan^{-1}(-\sqrt{3})$
Trigonometric Functions
tan y = -tan π/3
or tan y = tan (-π/3)
Since range of principle value of $\tan^{-1}$ is $\left[-\frac{\pi}{2}.\frac{\pi}{2}\right]$
Therefore, Principle value of $\tan^{-1}(-\sqrt{3})$ is -π/3.

Solution 5:

$\cos^{1}\left(\frac{-1}{2}\right)$
Y = $\cos^{-1}\left(\frac{-1}{2}\right)$
Cos y = -1/2
Cos y = $-\cos\frac{\pi}{3}$
Cos y = cos (∏ - ∏/3) = cos (2π/3)
Since principle value of $\cos^{-1}$ is [0. π]
Therefore, Principle value of $\cos^{-1}\left(\frac{-1}{2}\right)$ is 2π/3.

Solution 6:

$\tan^{-1}(-1)$
Let y = $\tan^{-1}(-1)$
Tan (y) = -1
Tan y = tan $\left(-\frac{\pi}{4}\right)$
Since principle value of $\tan^{-1}$ is $\left[-\frac{\pi}{2}.\frac{\pi}{2}\right]$
Therefore, Principle value of $\tan^{-1}(-1)$ is = π/4.

Solution 7:

$Sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$
Y = $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

Sec y = $2/\sqrt{3}$
Sec y = sec $\frac{\pi}{6}$
Since principle value of sec^{-1} is [0.π]
Therefore, Principle value of $sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ is π/6

Solution 8:

$cot^{-1}(\sqrt{3})$
Y = $cot^{-1}(\sqrt{3})$
Cot y = $\sqrt{3}$
Cot y = π/6
Since principle value of cot^{-1} is [0.π]
Therefore, Principle value of cot^{-1} $(\sqrt{3})$ is π/6.

Solution 9:

$cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$
Let y = $cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$
Cosy = $-\frac{1}{\sqrt{2}}$
Cos y = - $cos\frac{\pi}{4}$
Cos y = cos $\left[\pi - \frac{\pi}{4}\right]$ = cos $\frac{3\pi}{4}$
Since Principle value of cos^{-1} is [0.π]
Therefore. Principle value of $cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ is 3π/4.

Solution 10:

$cosec^{-1}$ $(-\sqrt{2})$
Ley = $cosec^{-1}$ $(-\sqrt{2})$
Cosec y = $-\sqrt{2}$
Cosec y = cosec $\frac{-\pi}{4}$
Since Principle value of $cosec^{-1}$ is $\left[-\frac{\pi}{2}.\frac{\pi}{2}\right]$
Therefore, Principle value of $cosec^{-1}$ $(-\sqrt{2})$ is – π/4

Find the values of the following:

11. $tan^{-1}(1) + cos^{-1} -\frac{1}{2} + sin^{-1} -\frac{1}{2}$
12. $cos^{-1}\frac{1}{2} + 2\ sin^{-1}\frac{1}{2}$
13. If sin^{-1} x = y, then
(A) 0 ≤ y ≤π
(B) $-\frac{\pi}{2} \leq y \frac{\pi}{2}$

(C) $0 < y < \pi$

(D) $-\frac{\pi}{2} \leq y \frac{\pi}{2}$

14. $\tan^{-1}(\sqrt{3}) - \sec^{-1}(-2)$ is equal to

(A) π

(B) $-\pi/3$

(C) $\pi/3$

(D) $2\pi/3$

Solution 11.

$\text{Tan}^{-1}(1) + \cos^{-1}\left(\frac{-1}{2}\right) + \sin^{-1}\left(\frac{-1}{2}\right)$

$= \tan^{-1} \tan\frac{\pi}{4} + \cos^{-1}\left(-\cos\frac{\pi}{3}\right) + \sin^{-1}\left(-\sin\frac{\pi}{6}\right)$

$= \frac{\pi}{4} + \cos\left(\pi - \frac{\pi}{3}\right) + \sin^{-1}\sin\left(-\frac{\pi}{6}\right)$

$= \frac{\pi}{4} + \frac{2\pi}{3} + \frac{\pi}{6}$

$= \frac{3\pi + 8\pi - 2\pi}{12}$

$= \frac{9\pi}{12} = \frac{3\pi}{4}$

Solution 12:

Let $\cos^{-1}\left(\frac{1}{2}\right) = x$. Then, $\cos x = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right)$

$\text{Cos}^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

Let $\sin^{-1}\left(\frac{1}{2}\right) = y$. Then, $\sin y - \frac{1}{2} = \sin\left(\frac{\pi}{6}\right)$

$\text{Sin}^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

Now,

$\text{Cos}^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} + \frac{2\pi}{6}$

$= \frac{\pi}{3} + \frac{\pi}{3}$

$= \frac{2\pi}{3}$

Solution 13.

Option (B) is correct.

Given $\sin^{-1} x = y$,

The range of the principle value of $\sin^{-1}$ is $\left[-\frac{\pi}{2}.\frac{\pi}{2}\right]$

Therefore, $-\frac{\pi}{2} \leq y \frac{\pi}{2}$

Solution 14.

Option (B) is correct.

$\text{Tan}^{-1}(\sqrt{3}) - \sec^{-1}(-2) = \tan^{-1}(\tan \pi/3) - \sec^{-1}(-\sec \pi/3)$

$= \pi/3 - \sec^{-1}(\sec(\pi/3))$

$= \pi/3 - 2\pi/3 = -\pi/3$

Exercise 2.2

Prove the following:

Question 1

$3 \sin^{-1} x = \sin^{-1} (3x - 4x^3)$. X $\in \left[-\frac{1}{2}, \frac{1}{2}\right]$

Solution:

$3 \sin^{-1} x = \sin^{-1} (3x - 4x^3)$, $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$
(Use identity: $\sin 3\theta = 3 \sin\theta - 4\sin^3\theta$)
Let x = $\sin\theta$. Then
$\theta = \sin^{-1} x$
Now, RHS
$= \sin^{-1} (3x - 4x^3)$
$= \sin^{-1} (3\sin\theta - 4\sin^3)$
$= \sin^{-1} (\sin 3\theta)$
$= 3\theta$
$= 3 \sin^{-1} x$
= LHS
Hence Proved

Question 2

$3\cos^{-1} x = \cos^{-1} (4x^3 - 3x)$, $x \in \left[\frac{1}{2}, 1\right]$

Solution:

$3\cos^{-1} = x = \cos^{-1} (4x^3 - 3x)$, $x \in \left[\frac{1}{2}, 1\right]$
Using identity: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$
Put $x = \cos\theta$
$\theta = \cos^{-1} (x)$
Therefore, $\cos 3\boldsymbol{\theta} = 4x^3 - 3x$
RHS:
$\cos^{-1} (4x^3 - 3x)$
$= 3\theta$
$= 3 \cos^{-1} (x)$
= LHS
Hence Proved.

Question 3

$\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$

Solution:

$\text{Tan}^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$

$\text{Tan}^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}$

Using identity:

LHS = $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24}$

$= \tan^{-1}\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}x\frac{7}{24}}$

$= \tan^{-1}\frac{48+77}{264-14}$

$= \tan^{-1}(125/250)$

$= \tan^{-1}(1/2)$

= RHS

Hence Proved

Question 4

$2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$

Solution:

$2\tan^{-1} x = \tan^{-1}\frac{2x}{1-x2}$

Use identity:

LHS

$= 2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7}$

$= \text{Tan}^{1}\frac{2x\frac{1}{2}}{1-(\frac{1}{2})} + \tan^{-1}\frac{1}{7}$

$= \tan^{-1}(4/3) + \tan^{-1}(1/7)$

Again, using identity:

$\text{Tan}^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}$

We have,

$\text{Tan}^{-1}\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}x\frac{1}{7}}$

$= \tan^{-1}(\frac{28+3}{21-4})$

$= \tan^{-1 (31/17)}$

RHS

write the following functions in the simplest form:

Question 5

$\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}, x \neq 0$

Solution:

Let's say x = tan θ then θ = $\tan^{-1}$ x
We get.

$\tan^{-1}\frac{\sqrt{1+x^2}-1}{x} = \tan^{-1}\left[\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right]$

$= \tan^{-1}\left\{\frac{\sec\theta-1}{\tan\theta}\right\}$

$= \tan^{-1}\left[\frac{1-\cos\theta}{\sin\theta}\right]$

$= \tan^{-1}\left\{\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right\}$

$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$

This is simplest form of the function.

Question 6

$\tan^{-1}\frac{1}{\sqrt{x^2-1}}, |x| > 1$

Solution:

Let us consider, x = sec θ, then θ = $\sec^{-1}$ x

$\tan^{-1}\frac{1}{\sqrt{x^2-1}} = \tan^{-1}\frac{1}{\sqrt{\sec^2\theta-1}}$

$= \tan^{-1}\frac{1}{\sqrt{\tan\theta}}$

$= \tan^{-1}\left[\frac{1}{\tan\theta}\right]$

$= \tan^{-1}(\cot\theta)$

$= \tan^{-1}\tan(\pi/2-\theta)$

$= (\pi/2-\theta)$

$= \pi/2 - \sec^{-1}x$

This is simplest form of the given function.

Question 7

$\tan^{-1}\left[\sqrt{\frac{1-\cos x}{1+\cos x}}\right]. 0 < x < \pi$

Solution:

$\mathrm{Tan}^{-1}\left[\sqrt{\frac{1-\cos x}{1+\cos x}}\right] = \tan^{-1}\left[\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}\right]$

$= \tan^{-1}\left[\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right]$

$= \tan^{-1}\left(\tan\frac{x}{2}\right) = \frac{x}{2}$

Question 8

$\text{Tan}^{-1}\left(\frac{cosx-sinx}{cosx+sinx}\right), \frac{-\pi}{4} < x < \frac{3\pi}{4}$

Solution:

Divide numerator and denominator by cos x, we have

$\tan^{-1}\left(\frac{\frac{coxs\ (x)}{\cos(x)}-\frac{\sin(x)}{\cos(x)}}{\frac{\cos(x)}{\cos(x)}-\frac{\sin(x)}{\cos(x)}}\right)$

$= \tan^{-1}\left(\frac{1-\frac{\sin(x)}{\cos(x)}}{1+\frac{\sin(x)}{\cos(x)}}\right)$

$\tan^{-1}\left(\frac{1-tanx}{1+tanx}\right)$

$\text{Tan}^{-1}\left[\frac{tan\frac{\pi}{4}-tanx}{1+tan\frac{\pi}{4}tanx}\right]$

$= \tan^{-1}\tan(\pi/4 - x)$

$= \pi/4 - x$

Question 9

$\tan^{-1}\frac{x}{\sqrt{a2-x2}}, |x| < a$

Solution:

Put x =a sin θ, which implies sin θ = x/a and $\theta = \sin^{-1}(x/a)$

Substitute the value into given function, we get

$\tan^{-1}\frac{x}{\sqrt{a2-x2}} = \tan^{-1}\left[\frac{asin\theta}{\sqrt{a^2-a^2sin^2\theta}}\right]$

$= \tan^{-1}\left[\frac{asin\theta}{a\sqrt{1-sin^2\theta}}\right]$

$= \tan^{-1}\left(\frac{asin\theta}{acos\theta}\right)$

$= \tan^{-1}(\tan\theta)$

$= \theta$

$= \sin^{-1}(x/a)$

Question 10

$\tan^{-1}\left[\frac{3a^2x-x^3}{a^3-3ax^2}\right], a > 0: \frac{-a}{\sqrt{3}} < x < \frac{a}{\sqrt{3}}$

Solution:

After dividing numerator and denominator by a^3 we have.

$\tan^{-1}\left(\frac{3\left(\frac{x}{a}\right)-\left(\frac{x}{a}\right)^3}{1-3\left(\frac{x}{a}\right)^2}\right)$

Put x/a = tan θ and $\theta = \tan^{-1}$ (x/a)
$\tan^{-1}\left[\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right]$
= $\tan^{-1}$ (tan 3 θ)
= 3 θ
= 3 $\tan^{-1}$(x/a)
Find the values of each of the following:

Question 11

$\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]$

Solution:
= $\tan^{-1}\left[2\cos\left(2\sin^{-1}\sin\frac{\pi}{6}\right)\right]$
= $\tan^{-1}\left[2\cos\left(2x\frac{\pi}{6}\right)\right]$
= $\tan^{-1}$ (2 cos π /3)
= $\tan^{-1}$ (2 x ½)
= $\tan^{-1}$ (1)
= $\tan^{-1}$ (tan (π /4))
= π /4

Question 12

cot ($\tan^{-1}$a + $\cot^{-1}$a)

Solution:

Cot ($\tan^{-1}$a + $\cot^{-1}$a) = cot π/2 = 0
Using identity: $\tan^{-1}$a + $\cot^{-1}$a = π/2

Question 13

$\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right]$, **|x| <1. Y > 0 and xy < 1**

Solution:

Put x = tan θ and y = tan ϕ, we have
$\tan\frac{1}{2}\left[\sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta} + \cos^{-1}\frac{1-\tan^2\phi}{1+\tan^2\phi}\right]$
= tan1/2 [$\sin^{-1}$ sin 2 θ + $\cos^{-1}$ cos 2 ϕ]
= tan (1/2) [2 θ + 2 ϕ]
= tan (θ + ϕ)
= $\frac{\tan\theta + \tan\phi}{1-\tan\phi\tan\phi}$
= (x+y) / (1 – xy)

Question 14

if sin $\left[\sin^{-1}\frac{1}{5} + \cos^{-1}x\right]$ = 1, then find the value of x.

Solution:

We know that, sin 90 degrees = sin π/2 = 1
So, given equation turned as,
$\sin^{-1}\frac{1}{5} + \cos^{-1} x = \frac{\pi}{2}$
$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1}\frac{1}{5}$
Using identity: $\sin^{-1} t + \cos^{-1} t = \pi/2$
$\cos^{-1} x = \cos^{-1}\frac{1}{5}$
we have,
which implies, the value of x is 1/5.

Question 15

if $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$, then find the value of x.

Solution:

We have reduced the given equation using below identity:
$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}$
$\tan^{-1}\frac{\frac{x-1}{x-2}+\frac{x+1}{z+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)} = \frac{\pi}{4}$
or
$\tan^{-1}\frac{(x-1)(x+2)+(x+1)(x-2)}{(x-2)(x+2)-(x-1)(x+1)} = \frac{\pi}{4}$
or
$\tan^{-1}\frac{x^2+2x-x-2+x^2-2x+x-2}{x^2-4-(x^2-1)} = \frac{\pi}{4}$
or $\frac{2x^2-4}{x^2-4-x^2+1} = \tan\left(\frac{\pi}{4}\right)$
or (2x^2 – 4)/-3 = 1
or 2x^2 = 1
or $x = \pm\frac{1}{\sqrt{2}}$
the value of x is either $\frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$

Find the values of each of the expressions in Exercise 16 to 18.

Question 16

$\sin^{-1}\left(\sin\left(\frac{2a}{3}\right)\right)$

Solution:

Given expression is $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$

First split $\frac{2\pi}{3}$ as $\frac{(3\pi-\pi)}{3}$ or $\pi - \frac{\pi}{3}$

After substituting in given we get.

$\text{Sin}^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \sin^{-1}\left(\sin\left(\pi - \frac{\pi}{3}\right)\right) = \frac{\pi}{3}$

Therefore, the value of $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$ is $\frac{\pi}{3}$

Question 17

$\tan^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right)$

Solution:

Given expression is $\tan^{-1}\left(\tan\left(\frac{3\pi}{3}\right)\right)$

First split $\frac{3\pi}{4}$ as $\frac{(4\pi-\pi)}{4}$ or $\pi - \frac{\pi}{4}$

After substituting in given we get.

$\tan^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right) = \tan^{-1}\left(\tan\left(\pi - \frac{\pi}{4}\right)\right) = -\frac{\pi}{4}$

The value of $\tan^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right)$ is $\frac{-\pi}{4}$.

Question 18

$\tan\left(\sin^{-1}\left(\frac{3}{5}\right) + \cot^{-1}\frac{3}{2}\right)$

Solution:

Given expression is $\tan\left(\sin^{-1}\left(\frac{3}{5}\right) + \cot^{-1}\frac{3}{2}\right)$

Putting, $\sin^{-1}\left(\frac{3}{5}\right) = x$ and $\cot^{-1}\left(\frac{3}{2}\right) = y$

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Or sin (x) = 3/5 and cot y = 3/2

Now, sin(x) = 3/5 => cos x = $\sqrt{1-\sin^2 x} = \frac{4}{5}$ and sec $x = 5/4$

(using identities: cos x = $\sqrt{1-\sin^2 x}$ and sec x = 1/cos x)

Again, tan x = $\sqrt{\sec^2 x - 1} = \sqrt{\frac{25}{10} - 1} =$ ¾ and tan y = 1/cot(y) = 2/3

Now, we can write given expression as,

$\text{Tan}\left(\sin^{-1}\left(\frac{3}{5}\right) + \cot^{-1}\frac{3}{2}\right) = \tan(x+y)$

$= \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}x\frac{2}{3}}$

= 17/6

Question 19

$\cos^{-1}(\cos\frac{7\pi}{6})$ is equal to
(A) 7 π/6 (B) 5 π/6 (C) π/3 (D) π/6

Solution:

Option (B) is correct.
Explanation:
$\cos^{-1}(\cos\frac{7\pi}{6}) = \cos^{-1}(\cos(2\pi - \frac{7\pi}{6})$
(As cos (2 π – A) = cos A)
Now $2\pi - \frac{7\pi}{6} = \frac{12\pi - 7\pi}{6} = \frac{5\pi}{6}$

Question 20

$\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$ is equal to
(A) ½ (B) 1/3 (C) ¼ (D) 1

Solution:

Option (D) is correct.
Explanation:
First solve for: $\sin^{-1}\left(-\frac{1}{2}\right)$
$\sin^{-1}\left(-\frac{1}{2}\right) = \sin^{-1}\left(-\sin\frac{\pi}{6}\right) = \sin^{-1}\left[\sin\left(-\frac{\pi}{6}\right)\right]$
= - π/6
Again,
$\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$
$= \sin\left[\frac{\pi}{3} - \left(-\frac{\pi}{6}\right)\right]$
$= \sin\left[\frac{\pi}{3} + \frac{\pi}{6}\right]$
= sin (π/2)
=1

Question 21

$\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})$ is equal to
(A) π (B) – π/2 (C) 0 (D) $2\sqrt{3}$

Solution:

Option (B) is correct.
Explanation:
$\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})$ can be written as

$= \tan^{-1} \tan\frac{\pi}{3} - \cot^{-1}\left(-cot\frac{\pi}{6}\right)$
$= \frac{\pi}{3} - \cot^{-1}\left[\cot\left(\pi - \frac{\pi}{6}\right)\right]$
$= \frac{\pi}{3} - (\pi - \frac{\pi}{6})$
$= \frac{\pi}{3} - \frac{5\pi}{6}$
$= \frac{-3\pi}{6}$
$= -\pi/2$

Miscellaneous Exercise

Find the value of the following:

Question 1

$\cos^{-1}(\cos\frac{13\pi}{6})$

Solution:

First solve for, $\cos\frac{13\pi}{6} = \cos(2\pi + \frac{\pi}{6}) = \cos\frac{\pi}{6}$
Now, $\cos^{-1}(\cos\frac{13\pi}{6}) = \cos^{-1}(\cos\frac{\pi}{6}) = \frac{\pi}{6} \in [0. \pi]$
[As $\cos^{-1}\cos(x) = x$ if $x \in [0, \pi]$]
So, the value of $\cos^{-1}(\cos\frac{13\pi}{6})$ is $\frac{\pi}{6}$,

Question 2

$\tan^{-1}(\tan\frac{7\pi}{6})$

Solution:

First solve for, $\tan\frac{7\pi}{6} = \tan(\pi + \frac{\pi}{6}) = \tan\frac{\pi}{6}$
Now: $\tan^{-1}(\tan\frac{7\pi}{6}) = \tan^{-1}(\tan\frac{\pi}{6}) = \frac{\pi}{6} \in (-\pi/2, \pi/2)$
[As $\tan^{-1}\tan(x) = x$ if $x \in (-\pi/2, \pi/2)$]
So, the value of $\tan^{-1}(\tan\frac{7\pi}{6})$ is $\frac{\pi}{6}$.

Question 3

Prove that $2\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{24}{7}$

Solution:

Step 1: Find the value of cos x and tan x

Let us consider $\sin^{-1}\frac{3}{5}$ = x, then sin x = 3/5

So, cos x = $\sqrt{1 - sin^2 x}$ = $\sqrt{1 - (\frac{3}{5})^2}$ = 4/5

tan x = sin x/ cos x = ¾

therefore, x = $\tan^{-1}$ (3/4), substitute the value of x,

$\sin^{-1}\frac{3}{5} = \tan^{-1}\left(\frac{3}{4}\right)$.... (1)

step 2: Solve LHS

2 $\sin^{-1}\frac{3}{5}$ = 2 $\tan^{-1}\frac{3}{4}$

Using identity =: 2$\tan^{-1}$ x = $\tan^{-1}$ $(\frac{2x}{1-x^2})$, we get

= $\tan^{-1}$ $(\frac{2(\frac{3}{4})}{1-(\frac{3}{4})^2})$

= $\tan^{-1}$ (24/7)

= RHS

Hence Proved.

Question 4

Prove that $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{36}$

Solution:

Let $\sin^{-1}$ $(\frac{8}{17})$ = x then sin x 8/17

Again, cos x = $\sqrt{1 - sin^2 x}$ = $\sqrt{1 - \frac{64}{289}}$ = 15/17

And,tan x = sin x / cos x = 8/ 15

Again,

Let $\sin^{-1}\left(\frac{3}{5}\right)$ = y then sin y = 3/5

Again, cos y = $\sqrt{1 - sin^2 y}$ = $\sqrt{1 - \frac{9}{25}}$ = 4/5

And tan y = sin y / cos y = ¾

Solve for tan (x + y), using below identity.

tan (x + y) = $\frac{tan\ x + tan\ y}{1 - tan\ x\ tan\ y}$

= $\frac{\frac{8}{15}+\frac{3}{4}}{1-\frac{8}{15}x\frac{3}{4}}$

= $\frac{32+45}{60-24}$

= 77/36

This implies x + y = $\tan^{-1}$ (77/36)

Substituting the values back, we have

$\sin^{-1}\frac{8}{17} + \sin^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$

Question 5

Prove that $\cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$

Solution:

Let $\cos^{-1}\frac{4}{5} = 0$ Let $\cos^{-1}\frac{12}{13} = \phi$

$\text{Cos}\ \theta = \frac{4}{5}$ $\cos\phi = \frac{12}{13}$

$\text{Sin}\ \theta = \sqrt{1-\cos^2\theta}$ $\sin\phi = \sqrt{1-\cos^2\phi}$

$= \sqrt{1-\frac{16}{25}}$ $= \sqrt{1-\frac{144}{169}}$

$= \frac{3}{5}$ $= \frac{5}{13}$

Solve the expression, Using identity: cos $(\theta + \phi) = \cos\theta \cos\phi - \sin\theta \sin\phi$
= 4/5 x 12/13 – 3/5 x 5/13
= (48 – 15)/65
= 33/65
This implies cos $(\theta + \phi)$ = 33/65
Or $\theta + \phi = \cos^{-1}$ (33/65)
Putting back the value of θ and ϕ, we get
$\text{Cos}^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$
Hence proved.

Question 6

Prove that $\cos^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right)$

Solution:

Let $\cos^{-1}\frac{12}{13} = \theta$ let $\sin^{-1}\frac{3}{5} = \phi$

So, $\cos\theta = \frac{12}{13}$ So $\sin\phi = \frac{3}{5}$

$\text{Sin}\theta = \sqrt{1-\cos^2\theta}$ $\cos\phi = \sqrt{1-\sin^2\phi}$

$= \sqrt{1-\frac{144}{169}}$ $= \sqrt{1-\frac{9}{25}}$

$= \frac{5}{13}$ $= \frac{4}{5}$

Solve the expression, Using identity: sin $(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi$
= 12/13 x 3/5 + 12/13 x 3/5
= (20 + 36)/65
= 56/65
Or sin $(\theta + \phi)$ = 56/65
Or $\theta + \phi) = \sin^{-1}$ 56/65
Putting back the value of θ and ϕ, we get
$\text{Cos}^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right)$
Hence Proved.

Question 7

Prove that $\tan^{-1}\left(\frac{63}{16}\right) = \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right)$

Solution:

Let $\sin^{-1}\frac{5}{13} = \theta$
So, $\sin\theta = \frac{5}{13}$
$\cos\theta = \sqrt{1 - \sin^2\theta}$
$= \sqrt{1 - \frac{25}{169}}$
$= \frac{12}{13}$
$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{5}{12}$

let $\cos^{-1}\frac{3}{5} = \phi$
so $\cos\phi = \frac{3}{5}$
$\sin\phi = \sqrt{1 - \cos^2\phi}$
$= \sqrt{1 - \frac{9}{25}}$
$= \frac{4}{5}$
$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{4}{3}$

Solve the expression, Using identity:
$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$
$= \frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}x\frac{4}{3}}$
= 63/16
$(\theta + \phi) = \tan^{-1}(63/16)$

Putting back the value of θ and ϕ, we get
$\tan^{-1}\left(\frac{63}{16}\right) = \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right)$
Hence Proved.

Question 8

Prove that $\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$

Solution:

LHS = $\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{8}\right))$
Solve above expressions, using below identity:

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$$

$= \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}x\frac{1}{7}}\right) + \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}x\frac{1}{8}}\right)$
After simplifying, we have
$= \tan^{-1}(6/17) + \tan^{-1}(11/23)$
Again, applying the formula, we get
After simplifying,
$= \tan^{-1}(325/325)$
$= \tan^{-1}(1)$

= π/4

Question 9

Prove that $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\frac{1-x}{1+x}$, x ∈ (0, 1)

Solution:

Let $\tan^{-1}\sqrt{x}$ =, then $\sqrt{x} = \tan\theta$
Squaring both the sides
$\tan^2\theta = x$
now, substitute the value of x in $\frac{1}{2}\cos^{-1}\frac{1-x}{1+x}$, we get
$= \frac{1}{2}\cos^{-1}\left[\frac{1-\tan^2\theta}{1+\tan^2\theta}\right]$
= ½ $\cos^{-1}(\cos 2\theta)$
½ (2θ)
$= \theta$
$= \tan^{-1}\sqrt{x}$

Question 10

Prove that $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}$, x ∈ (0, π/4)

Solution:

We can write 1+sin x as,
$1+\sin x = \cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\cos\frac{x}{2}\sin\frac{x}{2} = \left[\cos\frac{x}{2} + \sin\frac{x}{2}\right]^2$
And
$1+\sin x = \cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\cos\frac{x}{2}\sin\frac{x}{2} = \left[\cos\frac{x}{2} + \sin\frac{x}{2}\right]^2$
LHS:
$\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$
$= \cot^{-1}\left[\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)+\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)-\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}\right]$
$= \cot^{-1}\left(\frac{2\cos(\frac{x}{2})}{2\sin(\frac{x}{2})}\right)$
$= \cot^{-1}(\cot(x/2))$
= x/2

Question 11

Prove that $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x$, $-\frac{1}{\sqrt{2}} \leq x \leq 1$
[Hint: Put x = cos 2θ]

Solution:

Put x = cos 2θ so, $\theta = \frac{1}{2}\cos^{-1} x$

LHS = $\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$

$= \tan^{-1}\left|\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right|$

$= \tan^{-1}\left\{\frac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}\right\}$

$= \tan^{-1}\left\{\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}\right\}$

Divide each term by $\sqrt{2}\cos\theta$

$= \tan^{-1}\left\{\frac{1-\tan\theta}{1+\tan\theta}\right\}$

$= \tan^{-1}\left\{\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right\}$

$= \tan^{-1}\tan\left(\frac{\pi}{4}-\theta\right)$

$= \frac{\pi}{4}-\theta$

$= \frac{\pi}{4}-\frac{1}{2}\cos^{-1} x$

= RHS

Hence proved

Question 12

Prove that $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3} = \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$

Solution:

LHS = $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}$

$= \frac{9}{4}\left\{\frac{\pi}{2} = \sin^{-1}\frac{1}{3}\right\}$

$= \frac{9}{4}\cos^{-1}\frac{1}{3}$

.... (1)

(Using identity: $\sin^{-1}\theta + \cos^{-1}\theta = \frac{\pi}{2}$)

Let $\theta = \cos^{-1}(1/3)$, so $\cos\theta = 1/3$

As

Sin $\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{1}{9}} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$

Using equation (1). $\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$

Which is right hand side of the expression.

Solve the following equations:

Question 13

$2\tan^{-1}(\cos x) = \tan^{-1}(2 \text{ cosec } x)$

Solution:

$2\tan^{-1}(\cos x) = \tan^{-1}(2\text{cosec } x)$

$\tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}\left(\frac{2}{\sin x}\right)$

$\frac{2\cos x}{1-\cos^2 x} = \frac{2}{\sin x}$

$\frac{\cos x}{\sin x} = 1$

Cot x = 1

X = π/4

Question 14

Solve $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1} x, (x > 0)$

Solution:

Put $x = \tan\theta$

$\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1} x$

this implies

$\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1} x$

$\tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right) = \frac{1}{2}\tan^{-1}\tan\theta$

$\tan^{-1}\frac{\tan\frac{\pi}{4} - \tan\theta}{\tan\frac{\pi}{4} + \tan\theta} = \frac{1}{2}\theta$

$\tan^{-1}\tan\left(\frac{\pi}{4} - \theta\right) = \frac{\theta}{2}$

$\pi/4 - \theta = \theta/2$

or $3\theta/2 = \pi/4$

$\theta = \pi/6$

therefore, $x = \tan\theta = \tan \pi/6 = 1/\sqrt{3}$

Question 15

$\sin(\tan^{-1} x), |x| < 1$ is equal to

(A) $\frac{x}{\sqrt{1-x^2}}$ **(B)** $\frac{1}{\sqrt{1-x^2}}$

(c) $\frac{1}{\sqrt{1+x^2}}$ **(D)** $\frac{x}{\sqrt{1+x^2}}$

Solution:

Option (D) is correct.

Expiation:

Let $\theta = \tan^{-1} x$ so, $x = \tan\theta$

Again, let's say

Sin $(\tan^{-1}x) = \sin\theta$
This emplies,
Sin $(\tan^{-1} x) = \frac{1}{cosec\ \theta} = \frac{1}{\sqrt{1+cot^2\theta}}$
Put $\cot\theta = \frac{1}{tan\theta} = \frac{1}{x}$
Which shows,
Sin $(\tan^{-1}x) = \frac{1}{\sqrt{1+\frac{1}{x^2}}} = \frac{x}{\sqrt{x^2+1}}$

Question 16

$\sin^{-1}(1-x) - 2\sin^{-1} x = \frac{\pi}{2}$ then x is equal to
(A) 0, ½ (B) 1, ½ (C) 0 (D) ½

Solution:

Option (C) is correct.
Explanation:
Put $\sin^{-1} x = \theta$ so, $x = \sin\theta$
Now,
$\sin^{-1}(1-x) - 2\sin^{-1} x = \frac{\pi}{2}$
$\sin^{-1}(1-x) - 2\theta = \frac{\pi}{2}$
$\sin^{-1}(1-x) = \frac{\pi}{2} + 2\theta$
$1 - x = \sin\left(\frac{\pi}{2} + 2\theta\right)$
$1 = x = \sin\cos 2\theta$
$1 - x = 1 - 2x^2$
(As x = sin)
After simplifying, we get
X (2x – 1) = 0
X = 0 or 2x – 1 = 0
X = 0 or x = ½
Equation is not true for x = ½.so, the answer is x = 0.

Question 17

$\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$ is equal to
(A) π/2 (B) π/3 (C) π/4 (D) -3 π/4

Solution:

Option (C) is correct.
Explanation:

Given expression can be written as,

$= \tan^{-1}\left[\dfrac{\frac{x}{y}-\left(\frac{x-y}{x+y}\right)}{1+\frac{x}{y}\left(\frac{x-y}{x+y}\right)}\right]$

$= \tan^{-1}\left[\dfrac{x(x+y)-y(x-y)}{y(x+y)+x\,(x-y)}\right]$

$= \tan^{-1}\left\{\dfrac{x^2+xy-xy+y^2}{xy+y^2+x^2-xy}\right\}$

$= \tan^{-1}\left\{\dfrac{x^2+y^2}{x^2+y^2}\right\}$

$= \tan^{-1}(1)$

$= \pi/4$