Chapter 13
Limits And Derivatives

Exercise 13.1

Question 1

Evaluate the Given limit: $\lim_{x \to 3} x + 3$

Solution:

Given
$\lim_{x \to 3} x + 3$
Substituting x = 3, we get
= 3 + 3
= 6

Question 2

Evaluate the Given limit: $\lim_{x \to \pi} \left(x - \frac{22}{7}\right)$

Solution:

Given limit: $\lim_{x \to \pi} \left(x - \frac{22}{7}\right)$
Substituting x = π, we get
$\lim_{x \to \pi} \left(x - \frac{22}{7}\right)$= (π – 22 / 7)

Question 3

Evaluate the Given limit: $\lim_{r \to 1} r^2$

Solution:

Given limit: $\lim_{r \to 1} r^2$
Substituting r = 1, we get
$\lim_{r \to 1} r^2$ = π $(1)^2$
= π

Question 4

Evaluate the Given limit: $\lim_{x \to 4} \frac{4x+3}{x-2}$

Solution:

Given limit: $\lim_{x \to 4} \frac{4x+3}{x-2}$
Substituting x = 4, we get
$\lim_{x \to 4} \frac{4x+3}{x-2}$= [4(4) + 3] / (4 – 2)
= (16 + 3)/2
= 19 /32

Question 5

Evaluate the Given limit: $\lim_{x \to -1} \frac{x^{10}+x^5+1}{x-1}$

Solution:

Given limit: $\lim_{x \to -1} \frac{x^{10}+x^5+1}{x-1}$
Substituting x = -1, we get
$$\lim_{x \to -1} \frac{x^{10} + x^5 + 1}{x - 1}$$
= [(-1)10 + (-1)^5 + 1] / (-1 – 1)
= (1 – 1 + 1) / - 2
= - 1 / 2

Question 6

Evaluate the Given limit: $\lim_{x \to 0} \frac{(x+1)^5-1}{x}$

Solution:

Given limit: $\lim_{x \to 0} \frac{(x+1)^5-1}{x}$
= [(0 + 1)^5 – 1] / 0
=0
Since, this limit is undefined
Substitute x + 1 = y, then x = y – 1
$$\lim_{y \to 1} \frac{(y)^5 - 1}{y - 1}$$

$$\lim_{y\to1}\frac{(y)^5-1^5}{y-1}$$

We know that

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}= na^{n-1}$$

Hence

$$\lim_{y-1}\frac{(y)^5-1^5}{y-1}$$

= $5\,(1)^{5-1}$

= $5\,(1)^4$

= 5

Question 7

Evaluate the Given limit: $\lim_{x\to2}\frac{3x^2-x-10}{x^2-4}$

Solution:

By evaluating the limit at x = 2, we get

$\lim_{x\to2}\frac{3x^2-x-10}{x^2-4}$ = $[3\,(2)^2-x-10]/4-4$

= 0

Now, by factorizing numerator, we get

$$\lim_{x\to2}\frac{3x^2-x-10}{x^2-4}=\lim_{x\to2}\frac{3x^2-6x+5x-10}{x^2-2^2}$$

We know that

$a^2-b^2=(a-b)(a+b)$

= $\lim_{x\to2}\frac{(x-2)(3x+5)}{(x-2)(x+2)}$

= $\lim_{x\to2}\frac{(3x+5)}{(x+2)}$

By substituting x = 2, we get

= [3(2) +5] / (2+2)

= 11 / 4

Question 8

Evaluate the Given limit: $\lim_{x\to3}\frac{x^4-81}{2x^2-5x-3}$

Solution:

First substitute x = 3 in the given limit we get

$= \lim_{x \to 3} \frac{(3)^4 - 81}{2\,(3)^2 - 5 \times 3 - 3}$

= (81-81) / (18-8)

= 0 / 0

Since the limit is of the from 0/0, we need to factories the numerator and denominator

$= \lim_{x \to 3} \frac{(x^2-9)(x^2+9)}{2\,x^2-6x+x-3} = \lim_{x \to 3} \frac{(x-3)(x+3)(x^2+9)}{(2\,x+1)(x-3)}$

$= \lim_{x \to 3} \frac{x^4-81}{2\,x^2-5x-3} = \lim_{x \to 3} \frac{(x+3)(x^2+9)}{(2\,x+1)}$

Now substituting x = 3, we get

$= \frac{(3+3)(3^2+9)}{(2 \times 3+1)}$

= 108/7

= Hence

$$\lim_{x \to 3} \frac{x^4 - 81}{2x^2 - 5\,x - 3}$$

= 108 / 7

Question 9

Evaluate the Given limit: $\lim_{x \to 0} \frac{ax+b}{cx+1}$

Solution:

$$\lim_{x \to 0} \frac{ax + b}{cx + 1}$$

= [a (0) + b] / c (0) + 1

= b / 1

= b

Question 10

Evaluate the Given limit: $\lim_{z \to 1} \frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$

Solution:

$\lim_{z \to 1} \frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1} = (1 - 1) / (1 - 1)$

= 0

Let the value of $z^{1/6}$ be x

$\left(z^{1/6}\right)^2 x^2$

$z^{1/3} = x^2$

Now, substituting $z^{1/3} = x^2$ we get

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \frac{x^2 - 1^2}{x - 1}$$

We know that,

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

$$\lim_{x \to 1} \frac{x^2 - 1^2}{x - 1} = (1)^{2-1}$$

= 2

Question 11

Evaluate the Given limit: $\lim_{x \to 1} \frac{ax^2+bx+c}{cx^2+bx+a}, a + b + c \neq 0$

Solution:

Given limit: $\lim_{x \to 1} \frac{ax^2+bx+c}{cx^2+bx+a}, a + b + c \neq 0$

Substituting x = 1

$$\lim_{x \to 1} \frac{ax^2 + bx + c}{cx^2 + bx + a}$$

= [a $(1)^2$ + b (1) + c] / [c $(1)^2$ + b (1) + a]

= (a + b + c) / (a + b + c)

Given

[a + b + c ≠ 0]

= 1

Question 12

Evaluate the given limit: $\lim_{x \to -2} \frac{\frac{1}{x}+\frac{1}{2}}{x+2}$

Solution:

By substituting x = - 2, we get

$\lim_{x \to -2} \frac{\frac{1}{x}+\frac{1}{2}}{x+2}$ = 0/0

Now

$\lim_{x \to -2} \frac{\frac{1}{x}+\frac{1}{2}}{x+2} = \frac{\frac{2+x}{2x}}{x+2}$

= 1 / 2 x

= 1 /2 (-2)

= - 1/4

Question 13

Evaluate the Given limit: $\lim_{x \to 0} \frac{\sin ax}{bx}$

Solution:

Given $\lim_{x \to 0} \frac{\sin ax}{bx}$

Formula used here

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

By applying the limits in the given expression

$$\lim_{x \to 0} \frac{\sin ax}{bx} = \frac{0}{0}$$

By multiplying and dividing by `a' in the given expression, we get

$$\lim_{x \to 0} \frac{\sin ax}{bx} \times \frac{a}{a}$$

We get

$$\lim_{x \to 0} \frac{\sin ax}{ax} \times \frac{a}{b}$$

We know that

$$\lim_{x \to 0} \frac{\sin ax}{x} = 1$$

$$= \frac{a}{b} \lim_{ax \to 0} \frac{\sin ax}{ax} = \frac{a}{b} \times 1$$

= a/b

Question 14

Evaluate the given limit: $\lim_{x \to 0} \frac{\sin ax}{\sin bx}, a, b, \neq 0$

Solution:

$\lim_{x \to 0} \frac{\sin ax}{\sin bx} = 0/0$

By multiplying ax and bx in number and denominator, we get

$$\lim_{x \to 0} \frac{\sin ax}{\sin bx} = \lim_{x \to 0} \frac{\frac{\sin ax}{ax} \times ax}{\frac{\sin bx}{bx} \times bx}$$

Now we get

$$\frac{a}{b} \frac{\lim_{x \to 0} \frac{\sin ax}{ax}}{\lim_{x \to 0} \frac{\sin bx}{bx}}$$

We know that

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$
Hence, a/b × 1
= a /b

Question 15

Evaluate the Given limit: $\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}$

Solution:

$$\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}$$
$$\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)} = \lim_{\pi\to 0}\frac{\sin(\pi-x)}{\pi(\pi-x)}\times\frac{1}{\pi}$$
$$= \frac{1}{\pi}\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}$$
We know that
$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$
$$\frac{1}{\pi}\lim_{x\to 0}\frac{\sin(\pi-x)}{\pi(\pi-x)} = \frac{1}{\pi}\times 1$$
= 1 / π

Question 16

Evaluate the given limit: $\lim_{x\to 0}\frac{\cos x}{\pi-x}$

Solution:

$$\lim_{x\to 0}\frac{\cos x}{\pi-x} = \frac{\cos 0}{\pi-x}$$
= 1/ π

Question 17

Evaluate the given limit: $\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}$

Solution:

$$\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1} = \frac{0}{0}$$
Hence

$$\lim_{x\to0}\frac{\cos 2x-1}{\cos x-1}=\lim_{x\to0}\frac{1-2\sin^2 x-1}{1-2\sin^2\frac{x}{2}-1}$$

$(\cos 2x = 1 - 2\sin^2 x)$

$$\lim_{x\to0}\frac{\sin^2 x}{\sin^2\frac{x}{2}}=\lim_{x\to0}\frac{\frac{\sin^2 x\times x^2}{x^2}}{\frac{\sin^2\frac{x}{2}\times\frac{x^2}{4}}{\left(\frac{x}{2}\right)^2}}$$

$$= 4\,\frac{\lim_{x\to0}\frac{\sin^2 x}{x^2}}{\lim_{x\to0}\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}}$$

$$= 4\,\frac{\lim_{x\to0}\left(\frac{\sin^2 x}{x^2}\right)^2}{\lim_{x\to0}\left(\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)^2}$$

We know that,

$$\lim_{x\to0}\frac{\sin x}{x}=1$$

$= 4 \times 1^2 / 1^2$

$= 4$

Question 18

Evaluate the given limit: $\lim_{x\to0}\frac{ax+x\cos x}{b\sin x}$

Solution:

$$\lim_{x\to0}\frac{ax+x\cos x}{b\sin x}=\frac{0}{0}$$

Hence,

$$\lim_{x\to0}\frac{ax+x\cos x}{b\sin x}=\frac{1}{b}\lim_{x\to0}\frac{x(a+\cos x)}{\sin x}$$

$$=\frac{1}{b}\lim_{x\to0}\times\lim_{x\to0}(a+\cos x)$$

$$=\frac{1}{b}\times\frac{1}{\lim_{x\to0}\frac{\sin x}{x}}\times\lim_{x\to0}(a+\cos x)$$

We know that,

$$\lim_{x\to0}\frac{\sin x}{x}=1$$

$$=\frac{1}{b}\times(a+\cos 0)$$

$= (a + 1) / b$

Question 19

Evaluate the given limit: $\lim_{x\to 0} x \sec x$

Solution:

$$\lim_{x\to 0} x \sec x = \lim_{x\to 0} \frac{x}{\cos x}$$

$= \lim_{x\to 0} \frac{0}{\cos 0} = \frac{0}{1}$

$= 0$

Question 20

Evaluate the given limit: $\lim_{x\to 0} \frac{\sin ax + bx}{ax + \sin bx}$ $a, b, a + b \neq 0$

Solution:

$$\lim_{x\to 0} \frac{\sin ax + bx}{ax + \sin bx} = \frac{0}{0}$$

Hence,

$$\lim_{x\to 0} \frac{\sin ax + bx}{ax + \sin bx} = \lim_{x\to 0} \frac{\left(\sin \frac{ax}{ax}\right)\ ax + bx}{ax + \left(\sin \frac{bx}{bx}\right)}$$

$= \frac{\left(\lim_{ax\to 0} \sin \frac{ax}{ax}\right) \times \lim_{x\to 0} ax + \lim_{x\to 0} bx}{\lim_{x\to 0} ax + \lim_{x\to 0} bx \times \left(\lim_{bx\to 0} \sin \frac{bx}{bx}\right)}$

We know that,

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$= \frac{\lim_{x\to 0} ax + \lim_{x\to 0} bx}{\lim_{x\to 0} ax + \lim_{x\to 0} bx}$

We get

$= \frac{\lim_{x\to 0}(ax+bx)}{\lim_{x\to 0}(ax+bx)}$

$= 1$

Question 21

Evaluate the given limit: $\lim_{x\to 0}(\operatorname{cosec} x - \cot x)$

Solution:

$$\lim_{x\to 0}(\operatorname{cosec} x - \cot x)$$

Applying the formulas for cosec x and cot x, we get

Cosec x = $\frac{1}{\sin x}$ and $\cot x = \frac{\cos x}{\sin x}$

$\lim_{x\to 0}(\text{cosec}\,x - \cot x) = \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)$

$\lim_{x\to 0}(\text{cosec}\,x - \cot x) = \lim_{x\to 0}\frac{1-\cos x}{\sin x}$

Now by applying the formula we get,

$1 - \cos x = 2\sin^2\frac{x}{2}$ and $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

$\lim_{x\to 0}(\cos x - \cot x) = \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$

$\lim_{x\to 0}(\cos x - \cot x) = \lim_{x\to 0}\tan\frac{x}{2}$

= 0

Question 22

Evaluate the given limit: $\lim_{x\to\frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}}$

Solution:

$\lim_{x\to\frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}} = \frac{0}{0}$

Let x – (π/2) = y

Then x → (π/2) = y → 0

Now we get

$\lim_{x\to\frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}} = \lim_{y\to 0}\frac{\tan 2\left(y+\frac{\pi}{2}\right)}{y}$

$= \lim_{y\to 0}\frac{\tan(2y+\pi)}{y}$

$= \lim_{y\to 0}\frac{\tan(2y)}{y}$

We know that

tan x = sin x/ cos x

$= \lim_{y\to 0}\frac{\sin 2y}{y\cos 2y}$

By multiplying and dividing by 2, we get

$= \lim_{y\to 0}\frac{\sin 2y}{2y} \times \frac{2}{\cos 2y}$

$= \lim_{2y\to 0}\frac{\sin 2y}{2y} \times \lim_{y\to 0}\frac{2}{\cos 2y}$

= 1 × 2/ cos 0

= 1 × 2 / 1

= 2

Question 23

Find $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} 2x + 3 & x \le 0 \\ 3(x+1) & x > 0 \end{cases}$

Solution:

Given function is $f(x) = \begin{cases} 2x + 3 & x \le 0 \\ 3(x+1) & x > 0 \end{cases}$

$\lim_{x\to 0} f(x)$:

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0} (2x+3)$

= 2 (0) + 3

= 0 + 3

= 3

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0} 3(x+1)$

= 3 (0+1)

= 3 (1)

= 3

Hence,

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0} f(x) = 3$

Now, for $\lim_{x\to 1} f(x)$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} 3(x+1)$

= 3 (1+1)

= 3 (2)

= 6

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1} 3(x+1)$

= 3 (1+1)

= 3 (2)

= 6

Hence, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = 6$

$\lim_{x\to 0} f(x) = 3$ and $\lim_{x\to 1} f(x) = 6$

Question 24

Find $\lim_{x\to 1} f(x)$, where

$f(x) \begin{cases} x^2 - 1 & x \le 1 \\ -x^2 - 1 & x > 1 \end{cases}$

Solution:

Given function is:

$f(x) \begin{cases} x^2 - 1 & x \le 1 \\ -x^2 - 1 & x > 1 \end{cases}$

$\lim_{x \to 1} f(x)$:

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} x^2 - 1$

$= 1^2 - 1$

$= 1 - 1$

$= 0$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} (-x^2 - 1)$

$= (-1^2 - 1)$

$= -1 - 1$

$= -2$

We find

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} f(x)$

Hence, $\lim_{x \to 1^+} f(x)$ does not exist

Question 25

Evaluate $\lim_{x \to 0} f(x)$, where $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ x \\ 0, & x = 0 \end{cases}$

Solution:

Give function is $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ x \\ 0, & x = 0 \end{cases}$

We know that,

$\lim_{x \to a} f(x)$ exists only when $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

Now, we need to prove that: $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

We know

|x| = x, if x > =, -x, if x < 0

Hence,

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{|x|}{x}$

$\lim_{x \to 0} \frac{-x}{x} = \lim_{x \to 0} (-1)$

$= -1$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{|x|}{x}$

$\lim_{x \to 0} \frac{x}{x} = \lim_{x \to 0} (1)$

$= 1$

We find here,

$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

Hence, $\lim_{x \to 0} f(x)$ does not exist.

Question 26

Find $\lim_{x \to 0} f(x)$, where $f(x) = \begin{cases} \frac{x}{|x|}, x \neq 0 \\ x \neq 0 \\ 0 \end{cases}$

Solution:

$f(x) = \begin{cases} \frac{x}{|x|}, x \neq 0 \\ x \neq 0 \\ 0 \end{cases}$

$\lim_{x \to 0} f(x)$:

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{x}{|x|}$

$= \lim_{x \to 0} \frac{x}{-x} = \lim_{x \to 0} \frac{1}{-1}$

= - 1

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} \frac{x}{|x|}$

$\lim_{x \to 0} \frac{x}{x} = \lim_{x \to 0}(1)$

= 1

We find here,

$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

Hence, $\lim_{x \to 0} f(x)$ does not exist

Question 27

Find $\lim_{x \to 5} f(x)$, where $f(x) = |x| - 5$

Solution:

Given function is:

$f(x) = |x| - 5$

$\lim_{x \to 5} f(x)$:

$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} |x| - 5$

$\lim_{x \to 5}(x - 5) = 5 - 5$

= 0

$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} |x| - 5$

$= \lim_{x - 5}(x - 5)$

= 5 – 5

= 0

Hence, $\lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = \lim_{x \to 5} f(x) = 0$

Question 28

Suppose $f(x) = \begin{cases} a + bx, x < 1 \\ 4, \ x = 1 \\ b - ax \ x > 1 \end{cases}$ **and if** $\lim_{x \to 1} f(x) = f(1)$ **what are possible values of a and b**

Solution:

Given function is:

$f(x) = \begin{cases} a + bx, x < 1 \\ 4, \ x = 1 \\ b - ax \ x > 1 \end{cases}$ and

$\lim_{x \to 1} f(x) = f(1)$

$\lim_{x \to 1^-} f(x) \lim_{x \to 1} a + bx$

= a + b (1)

= a + b

$\lim_{x \to 1^+} f(x) \lim_{x \to 1} b - ax$

= b – a (1)

= b + a

Here

F(1) = 4

Hence, $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) \lim_{x \to 1} f(x) = f(1)$

Then, a + b = 4 and b – a = 4

By solving the above two equation we get

a = 0 and b = 4

Therefore, the possible values of a and b is 0 and 4 respectively

Question 29

Let $a_1, a_2 \dots \dots \dots a_n$ **be fixed real numbers an define a function f(x) =** $(x - a_1)(x - a_2) \dots \dots \dots \dots (x -$ **an.What is lim𝑥 →a1𝑓 𝑥?For some a ≠a1, a2………an, compute limx →af (x)**

Solution:

Given function is:

$f(x) = (x - a_1)(x - a_2) \dots \dots \dots (x - a_n)$

$\lim_{x \to a_1} f(x)$:

$\lim_{x \to a_1} f(x) = \lim_{x \to a_1} [(x - a_1)(x - a_2) \dots . \ (x - a_n)]$

$= [\lim_{x \to a_1}(x - a_1)][\lim_{x \to a_1}(x - a_2)] \dots \dots \dots [\lim_{x \to a_1}(x - a_n)]$

We get

$= (a_1 - a_1)(a_1 - a_2) \ldots \ldots . . (a_1 - a_n) = 0$
Hence, $\lim_{x \to a_1} f(x) = 0$
$\lim_{x \to a} f(x)$:
$\lim_{x \to a} f(x) = \lim_{x \to a}[(x - a_1)(x - a_2) \ldots . (x - a_n)]$
$= [\lim_{x \to a}(x - a_1)][\lim_{x \to a}(x - a_2)] \ldots \ldots \ldots [\lim_{x \to a}(x - a_n)]$
We get
$= (a - a_1)(a - a_2) \ldots \ldots . . (a - a_n)$
Hence, $\lim_{x \to a} f(x) = (a - a_1)(a - a_2) \ldots . (a - a_n)$
Therefore, $\lim_{x \to a_1} f(x) = 0$ and $\lim_{x \to a} f(x)(a - a_1)(a - a_2) \ldots . (a - a_n)$

Question 30

If $f(x) = \begin{cases} |x| + 1, & x < 0 \\ 0, & x = 0 \\ |x| - 1, & x > 0 \end{cases}$

Solution:

Given function is:
$f(x) = \begin{cases} |x| + 1, x < 0 \\ 0, \quad x = 0 \\ |x| - 1, x > 0 \end{cases}$
There three cases.
Case 1:
When a = 0
$\lim_{x \to 0} f(x)$:
$\lim_{x \to 0} f(x) = \lim_{x \to 0^-} (|x| + 1)$
$\lim_{x \to 0}(-x + 1) = -0 + 1$
= 1
$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (|x| + 1)$
$= \lim_{x \to 0}(x - 1) = 0 - 1$
= -1
Here, we find
$\lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^+} (|x| - 1)$
$= \lim_{x \to 0}(x - 1) = 0 - 1$
= - 1
Here, we find
$\lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^+} f(x)$
Hence, $\lim_{x \to 0} f(x)$ does not exit
Case: 2
When a <0

$\lim_{x \to a} f(x)$:

$\lim_{x \to a^-} f(x) = \lim_{x \to a^-} (|x| + 1)$

$\lim_{x \to a} (-x + 1) = -a + 1$

$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} (|x| + 1)$

$\lim_{x \to a} (-x + 1) = -a + 1$

Hence, $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a} f(x) = -a + 1$

Therefore, lim (f(x)) exists at x = a and a < 0

Case 3:

When a > 0

$\lim_{x \to a} f(x)$:

$\lim_{x \to a^-} f(x) = \lim_{x \to a^-} (|x| - 1)$

$\lim_{x \to a} (x - 1) = a - 1$

$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} (|x| - 1)$

$\lim_{x \to a} (x - 1) = a - 1$

Hence, $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a} f(x) = a - 1$

Therefore, lim (f(x)) exists at x = a and a > 0

Question 31

If the function f(x) satisfies $\lim_{x \to 1} \frac{f(x)-2}{x^2-1} = \pi$, evaluate $\lim_{x \to 1} f(x)$

Solution:

Given function that f (x) satisfies $\lim_{x \to 1} \frac{f(x)-2}{x^2-1} = \pi$

$\lim_{x \to 1} \frac{f(x) - 2}{x^2 - 1} = \pi$

$\lim_{x \to 1} (f(x) - 2) = \pi (\lim_{x \to 1} x^2 - 1)$

Substituting x = 1, we get

$\lim_{x \to 1} (f(x) - 2) = \pi (1^2 - 1)$

$\lim_{x \to 1} (f(x) - 2) = \pi (1 - 1)$

$\lim_{x \to 1} (f(x) - 2) = 0$

$\lim_{x \to 1} f(x) - \lim_{x \to 1} 2 = 0$

$\lim_{x \to 1} f(x) - 2 = 0$

= 2

Question 32

If $f(x)\begin{cases} mx^2 + n, x < 0 \\ nx + m, \ \ 0 \le x \le 1 \\ mx^2 + m. x > 1 \end{cases}$ for what integers m and n does both $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$exist?

Solution:

Given function is $f(x)\begin{cases} mx^2 + n, x < 0 \\ nx + m, \ \ 0 \le x \le 1 \\ mx^2 + m. x > 1 \end{cases}$

$\lim_{x\to 0} f(x)$:

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}(mx^2 + n)$

= m (0) + n

= 0 + n

= n

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0}(nx + m)$

= n (0) + m

= 0 + m

= m

Hence,

$\lim_{x\to 0} f(x)$exists if $n = m$

Now,

$\lim_{x\to 1} f(x)$:

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1}(nx + m)$

= n (1) + m

= n+ m

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1}(nx + m)$

= n $(1)^3$ + m

= n (1) + m

= n + m

Therefore $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x)$

Hence, for any integral value of m and n $\lim_{x\to 1} f(x)$ exists

Exercise 13.2

Question 1

Find the derivatives of $x^2 - 2$ at $x = 10$

Solution:

Let f (x) = $x^2 - 2$
From first principal
From first principal

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Put x = 10, we get

$f'(10) = \lim_{h \to 0} \frac{f(10+h) - f(10)}{h}$

$= \lim_{h \to 0} \frac{[(10+h)^2 - 2] - (10^2 - 2)}{h}$

$= \lim_{h \to 0} \frac{10^2 + 2 \times 10 \times h + h^2 - 2 - 10^2 + 2}{h}$

$= \lim_{h \to 0} \frac{20h + h^2}{h}$

$= \lim_{h \to 0} (20 + h)$

= 20 + 0

= 20

Question 2

Find the derivative of x at x = 1

Solution:

Let f(x) = x
Then,
From first principal

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Let f (x) = x

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(10)}{h}$

Put x = 1 we get

$f'(1) = \lim_{h \to 0} \frac{f(x+h) - f(1)}{h}$

$= \lim_{h \to 0} \frac{(1+h) - 1}{h}$

$= \lim_{h \to 0} \frac{1 + h - 1}{h}$

$= \lim_{h \to 0} \frac{h}{h}$

$= \lim_{h \to 0} 1$

= 1

Question 3

Find the derivative of 99x at x = 100.

Solution:

Let f (x) = 99x,
From first principle

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Put x = 100, we get

$$f'(100) = \lim_{h \to 0} \frac{f(100+h) - f(100)}{h}$$

$$= \lim_{h \to 0} \frac{99(100+h) - 99 \times 100}{h}$$

$$= \lim_{h \to 0} \frac{99 \times 100 + 99h - 99 \times 100}{h}$$

$$= \lim_{h \to 0} \frac{99 \times h}{h}$$

$$= \lim_{h \to 0} 99$$

= 99

Question 4

Find the derivative of the following functions from first principle
(i) $x^3 - 27$
(ii) (x – 1) (x – 2)
(iii) $1 / x^2$
(iv) x + 1 / x – 1

Solution:

(i) Let f (x) = $x^3 - 27$
From first principle

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[(x+h)^3 - 27] - (x^3 - 27)}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{h^3 + 3x^2h + 3xh^2}{h}$$

$$= \lim_{h \to 0} (h^3 + 3x^2h + 3xh^2)$$

$= 0 + 3x^2$

$= 3x^2$

(ii) Let f (x) = (x – 1) (x – 2)

From first principal

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h-1)(x+h-2) - (x-1)(x-2)}{h}$$

$$= \lim_{h \to 0} \frac{(x^2 + hx - 2x + hx + h^2 - 2h - x - h + 2) - (x^2 - 2x - x + 2)}{h}$$

$$= \lim_{h \to 0} \frac{hx + hx + h^2 - 2h - h}{h}$$

$$= \lim_{h \to 0}(h + 2x - 3)$$

$$= 0 + 2x - 3$$

$$= 2x - 3$$

(iii) Let f (x) = 1 /x^2

From first principal, we get

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x^2}{(x+h)^2} - \frac{1}{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{x^2 - (x+h)^2}{hx^2(x+h)^2}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{x^2 - x^2 - h^2\ 2hx}{x^2(x+h)^2}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{-h^2 - 2hx}{x^2(x+h)^2}\right]$$

$$= (0 - 2x)/\left[x^2(x+0)^2\right]$$

$$= (-2/x^3)$$

(iv) Let f (x) = x + 1 / x – 1

From first principle, we get

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x+h+1}{x+h-1} - \frac{x+1}{x-1}}{h}$$

$$= \lim_{h \to 0} \frac{(x-1)(x+h+1) - (x+1)(x+h-1)}{h(x-1)(x+h-1)}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{(x^2 + hx + x - x - h - 1) - (x^2 + hx + x - x + h - 1)}{(x-1)(x+h-1)}\right]$$

$$= \lim_{h \to 0} \frac{-2h}{h(x-1)(x+h-1)}$$

$$= \lim_{h \to 0} \frac{-2}{(x-1)(x+h-1)}$$

$= -\frac{2}{(x-1)(x-1)}$

$= -\frac{2}{(x-1)^2}$

Question 5

For the function $f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \cdots \frac{x^2}{2} + x + 1$ Prove that $f'(1) = 100 f, (0)$.

Solution:

Given function is:

$f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \cdots \frac{x^2}{2} + x + 1$

By differentiating both sides, we get

$\frac{d}{dx} f(x) = \frac{d}{dx}\left[\frac{x^{100}}{100} + \frac{x^{99}}{99} + \cdots \frac{x^2}{2} + x + 1\right]$

$= \frac{d}{dx}\left(\frac{x^{100}}{100}\right) + \frac{d}{dx}\left(\frac{x^{99}}{99}\right) + \cdots \ldots . + \frac{d}{dx}\left(\frac{x^2}{2}\right) + \frac{d}{dx}(x) + \frac{d}{dx}(1)$

We know that

$\frac{d}{dx}(x^n) = nx^{n-1}$

$\therefore \frac{d}{dx} f(x) = \frac{100\, x^{99}}{100} + \frac{99\, x^{98}}{99} + \cdots . + \frac{2x}{2} + 1 + 0$

$f'(x) = x^{99} + x^{98} + \cdots + x + 1$

At x = 0, we get

$f'(0) = 0 + 0 + \cdots + 0 + 1$

$f'(0) = 1$

At x = 1, we get

$f'(1) = 1^{99} + 1^{98} + \cdots . . +1 + 1 = [1 + 1 \ldots + 1]$ 100 times $= 1 \times 100 = 100$

Hence, $f'(1) = 100 f'(0)$

Question 6

Find the derivative of $x^n + ax^{n-1} + a^2x^{n-2} + \cdots . . + a^{n-1}x + a^n$ for some fied real number a.

Solution:

Given function is;

$f(x) = x^n + ax^{n-1} + a^2x^{n-2} + \cdots + a^{n-1}x + a^n$

By differentiating both sides, we get

$$f'(x) = \frac{d}{dx}(x^n + ax^{n-1} + a^2x^{n-2} + \cdots.. + a^{n-1}x + a^n)$$
$$= \frac{d}{dx}(x^n) + a\frac{d}{dx}(x^{n-1}) + a^2\frac{d}{dx}(x^{n-2}) + \cdots. + a^{n-1}\frac{d}{dx}(x) + a^{n-1}\frac{d}{dx}(1)$$
We know that
$$\frac{d}{dx}(x^n) = nx^{n-1}$$
$$f'(x) = nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \cdots ... + a^{n-1}a^n\ (0)$$
$$f'(x) = nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \cdots ... + a^{n-1}a^n$$

Question 7

For some constants a and b, find the derivative of
(i) (x – a) (x – b)
(ii) $(ax^2 + b)^2$
(iii) x – a / x – b

Solution:

(i) (x – a) (x – b)
Let f (x) = (x -a) (a-b)
F (x) = $x^2 - (a+b)x + ab$
Now, by differentiating both sides we get
$b^2(x^2 - (a+b)x + ab)$
$$= \frac{d}{dx}(x^2) - (a+b)\frac{d}{dx}(x) + \frac{d}{dx}(ab)$$
We know that
$$= \frac{d}{dx}(x^n) = nx^{n-1}$$
$$f'(x) = 2x - (a+b) + 0$$
(ii) $(ax^2 + b)^2$
Let f(x) = $a^2x^4 + 2ax^2 + b^2$
By differentiating both sides we get
$$f'(x) = \frac{d}{dx}(x^4) + (2ab)\frac{d}{dx}(x^2) + \frac{d}{dx}(b^2)$$
We know that
$$\frac{d}{dx}(x^n) = nx^{n-1}$$
$$f'(x) = a^2 \times 4x^2 + 2ab \times 2x + 0$$
$$= 4a^2x^3 + 4abx$$
$$= 4ax(ax^2 + b)$$
(iii) x – a/x – b
Let f (x) = $\frac{(x-a)}{(x-b)}$

By differentiating both sides and using quotient rule, we get

$$f'(x) = \frac{d}{dx}\left(\frac{x-a}{x-b}\right)$$

$$f'(x) = \frac{(x-b)\frac{d}{dx}(x-a) - (x-a)\frac{d}{dx}(x-b)}{(x-b)^2}$$

$$= \frac{(x-b)(1) - (x-a)(1)}{(x-b)^2}$$

By further calculation, we get

$$= \frac{x-b-x+a}{(x-b)^2}$$

$$= \frac{a-b}{(x-b)^2}$$

Question 8

Find the derivative of $\frac{x^n - a^n}{x-a}$ for some constant a.

Solution:

Let $f(x)\frac{x^n - a^n}{x-a}$

By differentiating both sides and using quotient rule we, get

$$f'(x) = \frac{d}{dx}\left(\frac{x^n - a^n}{x-a}\right)$$

$$f'(x) = \frac{(x-a)\frac{d}{dx}(x^n - a^n)\frac{d}{dx}(x-a)}{(x-a)^2}$$

By further calculation we get

$$= \frac{(x-a)(anx^{n-1} - 0) - (x^n - a^n)}{(x-a)^2}$$

$$= \frac{nx^n - anx^{n-1} - x^9 + a^n}{(x-a)^2}$$

Question 9

Find the derivative of

(i) 2x – 3 / 4

(ii) $(5x^3 + 3x - 1)(x - 1)$

(iii) $x^{-3}(5 + 3x)$

(iv) $x^5(3 - 6x^{-9})$

(v) $x^{-4}(3 - 4x^{-5})$

(vi) (2 / x + 1) – x^2 / 3x – 1

Solution:

(i)

Let f(x) 2x – 3 / 4

By differentiating both sides we get

$f'(x) = \frac{d}{dx}\left(2x - \frac{3}{4}\right)$

$= 2\frac{d}{dx}(x) - \frac{d}{dx}\left(\frac{3}{4}\right)$

= 2 – 0

= 2

(ii)

Let f (x) = $(5x^3 + 3x - 1)(x - 1)$

By differentiating both sides and using the product rule, we get

$f'(x) = (5x^3 + 3x - 1)\frac{d}{dx}(x - 2) + (x - 1)\frac{d}{dx}(5x^3 + 3x + 1)$

$= (5x^3 + 3x - 1) \times 1 + (x - 1) \times (15x^2 + 3)$

$(5x^3 + 3x + 1) + (x - 1)(15x^2 + 3)$

$= 5x^3 + 3x - 1 + 15x^3 + 3x - 15x^2 - 3$

$= 20x^3 - 15x^2 + 6x - 4$

(iii)

Let f (x) $x^{-3}(5 + 3x)$

By differentiating both sides and using Leibnitz product rule we get

$f'(x) = x^{-3}\frac{d}{dx}(5 + 3x) + (5 + 3x)\frac{d}{dx}(x^{-3})$

$= x^{-3}(0 + 3) + (5 + 3x)(-3x^{-3-1})$

By further calculation, we get

$= x^{-3}(3) + (5 + 3x)(-3x^{-4})$

$= 3x^{-3} - 15x^{-4} - -9x^{-3}$

$= -6x^{-3} - 15x^{-4}$

$= -3x^{-3}\left(2 + \frac{5}{x}\right)$

$= \frac{-3x^{-3}}{x}(2x + 5)$

(iv)

Let f (x) = $x^5(3 - 6x^{-9})$

By differentiating both sides and using Leibnitz product rule, we get

$f'(x) = x^5\frac{d}{dx}(3 - 6x^{-9}) + (3 - 6x^{-9})\frac{d}{dx}(x^5)$

$= x^5\{0 - 6(-9)x^{-9-1}\} + (3 - 6x^{-9})(5x^4)$

By further calculation we get

$= x^5(54x^{-10}) + 15x^4 - 30x^{-5}$

$= 54x^{-5} + 15x^4 - 30x^4$

$= 24x^{-5} + 15x^4$

$= 15x^4 + \frac{24}{x^5}$

(v)

Let f (x) = $x^{-4}(3 - 4x^{-5})$

By differentiating both sides and using Leibnitz product rule, we get

$f'(x) = x^{-4}\frac{d}{dx}(3 - 4x^{-5}) + (3 - 4x^{-5})\frac{d}{dx}(x^{-4})$

$= x^{-4}\{0 - 4(-5)x^{-5-1}\} + (3 - 4x^{-5})(-4)x^{-4-1}$

By further calculation, we get

$= x^{-4}(20x^{-6}) + (3 - 4x^{-5})(-4x^{-5})$

$= 20x^{-10} - 12x^{-5} + 16x^{-10}$

$= 36x^{-10} - 12x^{-5}$

$= -\frac{12}{x^5} + \frac{36}{x^{10}}$

(vi)

(2 / x + 1) – x ² / 3x – 1

Let $f(x) = \frac{2}{x-1} - \frac{x^2}{3x-1}$

By differentiating both sides we get,

$f'(x) = \frac{d}{dx}\left(\frac{2}{x-1} - \frac{x^2}{3x-1}\right)$

Using quotient rule we get

$f'(x)\left[\frac{(x+1)\frac{d}{dx}(2) - 2\frac{d}{dx}(x+1)}{(x+1)^2}\right] - \left[\frac{(3x-1)\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(3x-1)}{(3x-1)^2}\right]$

$= \left[\frac{(x+1)(0) - 2(1)}{(x+1)^2}\right] - \left[\frac{(3x-1)(2x) - (x^2) \times 3}{(3x-1)^2}\right]$

$= -\frac{2}{(x+1)^2} - \frac{x(3x-2)}{(3x-1)^2}$

Question 10

Find the derivative of cos x from first principle

Solution:

Let f (x) = cos x

Accordingly, f (x + h) = cos (x + h)

By first principal, we get

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

So, we get

$$= \lim_{h \to 0} \frac{1}{h}[\cos(x+h) - \cos(x)]$$
$$= \lim_{h \to 0} \frac{1}{h}\left[-2\sin\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)\right]$$
By further calculation we get
$$= \lim_{h \to 0} \frac{1}{h}\left[-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$
$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{2x+h}{2}\right) \times \lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}$$
$$= -\sin\left(\frac{2x+0}{2}\right) \times 1$$
$$= -\sin(2x/2)$$
$$= -\sin(x)$$

Question 11

Find the derivative of the following functions:
(i) sin x cos x
(ii) sec x
(iii) 5 sec x + 4 cos x
(iv) cosec x
(v) 3 cot x + 5 cosec x
(vi) 5 sin x – 6 cos x + 7
(vii) 2 tan x – 7 sec x

Solution:

(i) sin x cos x
Let f (x) = sin x cos x
Accordingly, from the first principal
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{\sin(x+h)\cos(x+h) - \sin x \cos x}{h}$$
$$= \lim_{h \to 0} \frac{1}{2h}[2\sin(x+h)\cos(x+h) - 2\sin x \cos x]$$
$$= \lim_{h \to 0} \frac{1}{2h}[\sin 2(x+h) - \sin 2x]$$
$$= \lim_{h \to 0} \frac{1}{2h}\left[2\cos\frac{2x+2h+2x}{2} . \sin\frac{2x+2h-2x}{2}\right]$$
By further calculation, we get
$$= \lim_{h \to 0} \frac{1}{h}\left[\cos\frac{4x+2h}{h}\sin\frac{2h}{h}\right]$$

$= \lim_{h \to 0} \frac{1}{h}[\cos(2x+h)\sin h]$

$= \lim_{h \to 0} \cos(2x+h). \lim_{h \to 0} \frac{\sin h}{h}$

$= \cos(2x+0).1$

$= \cos 2x$

(ii) sec x

Let df (x) = sec x

= 1/ cos x

By differentiating both sides, we get

$f'(x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$

Using quotient rule, we get

$f'(x) = \frac{\cos x \frac{d}{dx}(1) - 1 \frac{d}{dx}(\cos x)}{\cos^2 x}$

$= \frac{\cos x \times 0 - (-\sin x)}{\cos^2 x}$

We get

$= \frac{\sin x}{\cos^2 x}$

$= \frac{\sin x}{\cos x} \times \frac{1}{\cos x}$

= tan x sec x

(iii) 5 sec x + 4 cos x

Let f (x) = 5 sex x + 4 cos x

By differentiating both sides we get

$f'(x) = \frac{d}{dx}(5\sec x + 4\cos x)$

By further calculation, we get

$= 5\frac{d}{dx}(\sec x) + 4\frac{d}{dx}(\cos x)$

= 5 sec x tan x + 4 × (- sin x)

= 5 sec x tan × - 4 sin x

(iv) cosec x

Let f(x) = cosec x

Accordingly, f (x + h) = cosec (x + h)

By first principal, we get

f’ (x) = $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{\text{cosec}(x+h) - \text{cosec } x}{h}$

$= \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{\sin(x+h)} - \frac{1}{\sin x}\right)$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x - \sin(x+h)}{\sin(x+h)}\right]$$

$$= \frac{1}{\sin x}\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)}\right]$$

$$= \frac{1}{\sin x}\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)}\right]$$

By further calculation, we get

$$= \frac{1}{\sin x}\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\left(\frac{h}{2}\right)\sin(x+h)}\right]$$

$$= -\frac{1}{\sin x}\lim_{h\to 0}\frac{1}{h}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \times \lim_{h\to 0}\frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}$$

$$= -\frac{1}{\sin x}\times 1 \times \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+0)}$$

$$= -\frac{1}{\sin x}\times\frac{\cos x}{\sin x}$$

(v) 3 cot x + 5 cosec x

Let f (x) = 3 cot x 5 cosec x

F (x)' = (Cot x)' + 5 (cosec x)'

Let $f_1(x) = \cot$

Accordingly, $f_1(x+h) = \cot(x+h)$

By using first principal, we get

$$f_1(x) = \lim_{x\to 0}\frac{f_1(x+h) - f_1(x)}{h}$$

$$= \lim_{h\to 0}\frac{\cot(x+h) - \cot x}{h}$$

$$= \lim_{h\to 0}\left(\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x}\right)$$

By further calculation, we get

$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{\sin x\cos(x+h) - \cos x\sin(x+h)}{\sin x\sin(x+h)}\right)$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{\sin(x-x-h)}{\sin x\sin(x+h)}\right)$$

$$= 1/\sin x \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(-h)}{\sin(x+h)}\right]$$

$$= \frac{1}{\sin x}\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{1}{\sin(x+h)}\right)$$

$= -\frac{1}{\sin x} \times 1 \times \frac{1}{\sin(x+0)}$

$= -\frac{1}{\sin^2 x}$

$= \text{cosec}^2 \text{ x}$

Let f_2(x) = cosec x,

Accordingly, $f_2(x+h) = \cot(x+h)$

By using first principal, we get

$f'_2(x) = \lim_{x \to 0} \frac{f_2(x+h) - f_2(x)}{h}$

$= \lim_{h \to 0} \frac{\text{cosec}(x+h) - \text{cosec } x}{h}$

$= \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{\sin(x+h)} - \frac{1}{\sin x}\right)$

$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\sin x - \sin(x+h)}{\sin x \sin(x+h)}\right]$

By further calculation, we get

$= \frac{1}{\sin x} \lim_{h \to 0} \frac{1}{h}\left[\frac{2\cos\left(\frac{c=x+x+h}{h}\right)\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)}\right]$

$= \frac{1}{\sin x} \lim_{h \to 0} \frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\sin(x+h)}\right]$

$= \frac{1}{\sin x} \lim_{h \to 0}\left[\frac{-\sin\left(\frac{h}{2}\right)\cos\left(\frac{2x+h}{2}\right)}{\left(\frac{h}{2}\right)\sin(x+h)}\right]$

$= -\frac{1}{\sin x} \lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \times \lim_{h \to 0} \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}$

$= -\frac{1}{\sin x} \times 1 \times \frac{\cos\left(\frac{2x+0}{2}\right)}{\sin(x+0)}$

$= -\frac{1}{\sin x} \times \frac{\cos x}{\sin x}$

$= -\text{cosec } x \cot x$

Now, substitute the value of $(\cot x)'$ and $(\text{cosec})'$ in f′ (x) we get

f′(x) = 3 (cot x)′ + 5 (cosec x)′

f′(x) = 3 × $(-\text{cosec}^2 x) + 5 \times (-\text{cosec } x \cot x)$

$f'(x) = -3\,\text{cosec}^2 x - 5\,\text{cosec} \cot x$

(vi) 5 sin x – 6 cos x + 7

Let f (x) = 5 sin x – 6 cos x + 7

Accordingly, from the first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{1}{h}[2\tan(x+h) - 7\sec(x+h) - 2\tan x + 7\sec x]$$
$$= \lim_{h \to 0} \frac{1}{h}[2\{\tan(x+h) - \tan x\} - 7\{\sec(x+h) - \sec x\}]$$
$$= 2\lim_{h \to 0} \frac{1}{h}[\tan(x+h) - \tan x] - 7\lim_{h \to 0} \frac{1}{h}[\sec(x+h) - \sec x]$$

By further calculation, we get

$$= 2\lim_{h \to 0} \frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}\right] - 7\lim_{h \to 0} \frac{1}{h}\left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x}\right]$$
$$= 2\lim_{h \to 0} \frac{1}{h}\left[\frac{\sin(x+h)\cos x - \sin x\cos(x+h)}{\cos x\cos(x+h)}\right] - 7\lim_{h \to 0} \frac{1}{h}\left[\frac{\cos x - \cos(x+h)}{\cos x\cos(x+h)}\right]$$
$$= 2\lim_{h \to 0} \frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos x\cos(x+h)}\right] - 7\lim_{h \to 0}\left[\frac{-2\sin\left(\frac{x-x-h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos x\cos(x+h)}\right]$$

Now we get

$$= 2\lim_{h \to 0}\left[\left(\frac{\sin h}{h}\right)\frac{1}{\cos x\cos(x+h)}\right] - 7\lim_{h \to 0} \frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos x\cos(x+h)}\right]$$
$$= 2\left(\lim_{h \to 0} \frac{\sin h}{h}\right)\left(\lim_{h \to 0} \frac{1}{\cos x\cos(x+h)}\right) - 7\left(\lim_{\frac{h}{2} \to 0} \frac{\sin\frac{1}{h}}{\frac{h}{2}}\right)\left(\lim_{h \to 0} \frac{\sin\left(\frac{2x+h}{2}\right)}{\cos x\cos(x+h)}\right)$$
$$= 2.1\,\frac{1}{\cos x\cos x} - 7.1\left(\frac{\sin x}{\cos x\; cos}\right)$$
$$= 2\sec^2 x - 7\sec x\tan x$$

Miscellaneous exercise

Question 1

Find the derivative of the following functions from first principle:

(i) –x

(ii) (–x) $^{-1}$

(iii) sin (x + 1)

(iv) cos $\left(x - \frac{\pi}{8}\right)$

Solution:

(i) –x

Let f (x) = - x

Accordingly, f (x + h) = - (x + h)
Using first principal, we get
$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$,

$= \lim_{h \to 0} \frac{-(x+h)-(-x)}{h}$

Now we get

$= \lim_{h \to 0} \frac{-x-h+x}{h}$

$= \lim_{h \to 0} \frac{-h}{h}$

$= \lim_{h \to 0} (-1) = -1$

(ii) $(-x)^{-1}$

Let f (x) = $(-x)^{-1} = \frac{1}{-x} = \frac{-1}{x}$

Accordingly, f $(x+h) = \frac{-1}{(x+h)}$

Using first principal we get

$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h \to 0} \frac{-(x+h)-(-x)}{h}$

Now, we get

$= \lim_{h \to 0} \frac{-x-h+x}{h}$

$= \lim_{h \to 0} \frac{-h}{h}$

$= \lim_{h \to 0} (-1) = -1$

(ii) $(-x)^{-1}$

Let f (x) = $(-x)^{-1} = \frac{1}{-x} = \frac{-1}{x}$

Accordingly, f (x + h) $= \frac{-1}{(x+h)}$

Using first principal, we get

$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h \to 0} \frac{1}{h}\left[\frac{-1}{x+h} - \left(\frac{-1}{x}\right)\right]$

$= \lim_{h \to 0} \frac{1}{h}\left[\frac{-1}{x+h} + \frac{1}{x}\right]$

$= \lim_{h \to 0} \frac{1}{h}\left[\frac{-x+(x+h)}{x(x+h)}\right]$

By further calculation we get,

$= \lim_{h \to 0} \frac{1}{h}\left[\frac{-x+x+h}{x(x+h)}\right]$

$= \lim_{h \to 0} \frac{1}{h}\left[\frac{h}{x\,(x+h)}\right]$

$= \lim_{h \to 0} \frac{1}{x\,(x+h)}$

$= \frac{1}{x.x}$

(iii) sin (x + 1)

Let f (x) = sin (x+1)

Accordingly, f (x + h) = sin (x+h+1)

By using first principal, we get

f' (x) = $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h \to 0} \frac{1}{h}[\sin(x+h+1) - \sin(x+1)]$

$= \lim_{h \to 0} \frac{1}{h}\left[2\cos\left(\frac{x+h+1+x+1}{2}\right)\sin\left(\frac{x+h+1-x-1}{2}\right)\right]$

$= \lim_{h \to 0} \frac{1}{h}\left[2\cos\left(\frac{2x+h+2}{2}\right)\sin\left(\frac{h}{2}\right)\right]$

$= \lim_{h \to 0} \left[\cos\left(\frac{2x+h+2}{2}\right).\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]$

We get

$= \lim_{h \to 0} cos\left(\frac{2x+h+2}{2}\right).\lim_{\frac{h}{2} \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$

We know that

$h \to 0 \Rightarrow \frac{h}{2} \to 0$

$= \cos\left(\frac{2x+h+2}{2}\right).1$

= cos (x+1)

(iv) $\cos\left(x - \frac{\pi}{8}\right)$

Let f (x) = $\cos\left(x - \frac{\pi}{8}\right)$

Accordingly, f (x + h) = $\cos\left(x + h - \frac{\pi}{8}\right)$

By using first principal, we get

f' (x) $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h \to 0} \frac{1}{h}\left[cos\left(x + h - \frac{\pi}{8}\right) - cos\left(x - \frac{\pi}{8}\right)\right]$

We get

$$= \lim_{h \to 0} \frac{1}{h}\left[-2\sin\frac{\left(x+h-\frac{\pi}{8}-x+\frac{\pi}{8}\right)}{2}\sin\left(\frac{x+h-\frac{\pi}{8}-x+\frac{\pi}{8}}{2}\right)\right]$$

Further we get

$$= \lim_{h \to 0}\left[-\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\sin\frac{h}{2}\right]$$

$$= \lim_{h \to 0}\left[-\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\right].\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$\left[\text{As h } \to 0 \Rightarrow \frac{h}{2} \to 0\right]$$

$$= -\sin\left(\frac{2x+0-\frac{\pi}{4}}{2}\right).1$$

Hence, we get

$$= -\sin\left(x-\frac{\pi}{8}\right)$$

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers):

Question 2

(x + a)

Solution:

Let f (x) = x + a

Accordingly, f (x + h) = x + h + a

Using first principal, we get

$$f'(x) = \lim_{h \to 0}\frac{f(x+h)-f(x)}{h}$$

So, now we get

$$= \lim_{h \to 0}\frac{x+h+a-x-a}{h}$$

$$= \lim_{h \to 0}\left(\frac{h}{h}\right)$$

$$= \lim_{h \to 0}(1)$$

= 1

Question 3

(px + q) (r / x + s)

Solution:

$f'(x) = (px+q)\left(\frac{r}{x}+s\right)$
Using Leibnitz product rule, we get
$f'(x) = (px+q)\left(\frac{r}{x}+ s\right) + \left(\frac{r}{x}+ s\right)(px+q)'$
We get
$= (px+q)(rx^{-1}+s) + \left(\frac{r}{x}+ s\right)\ (p)$
By further calculation, we get
$= (px+q)(rx^{-2}) + \left(\frac{r}{x}+s\right)\ p$
$= (px+q)\left(\frac{-r}{x^2}\right) + \left(\frac{r}{x}+s\right)\ p$
Now, we get
$= \frac{-pr}{x} - \frac{qr}{x^2} + \frac{pr}{x} + ps$
$= ps - \frac{qr}{x^2}$

Question 4

(ax + b) (cx + d) ²

Solution:

Let f(x) = (ax +b) (cx + d) ²
By using Leibnitz product rule, we get
$f'(x) = \ (ax+b)\frac{d}{dx}(cx+d)^2 + \ (cx+d)^2\frac{d}{dx}(ax+b)$
We get,
$= \ (ax+b)\frac{d}{dx}(c^2x^2 + 2cdx + \ d^2) + \ (cx+d)^2\frac{d}{dx}(ax+b)$
By differentiating separately, we get
$= \ (ax+b)\left[\frac{d}{dx}(c^2x^2) + \frac{d}{dx}(2\ cdx) + \frac{d}{dx}d^2\right] + \ (cx+d)^2\left[\frac{d}{dx}\ ax + \frac{d}{dx}\ b\right]$
So,
$= (\mathrm{ax+b})(2\mathrm{c}^2\ \mathrm{x} + 2\ \mathrm{cd}) + \ (\mathrm{cx} + \ \mathrm{d}^2)\ \mathrm{a}$
$= 2c\ (ax+b)(c\ x+d) + a\ (cx+d)^2$

Question 5

(ax + b) / (cx + d)

Solution:

Let $f(x) = \frac{ax+b}{cx+d}$

Using quotient rule, we get

$$f'(x) = \frac{(cx+d)\frac{d}{dx}(ax+b) - (ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2}$$

Further we get

$$= \frac{(cx+d)(a) - (ax+b)(c)}{(cx+d)^2}$$

So, now we get

$$= \frac{acx + ad - acx - bc}{(cx+d)^2}$$

Hence,

$$= \frac{ad - bc}{(cx+d)^2}$$

Question 6

$(1 + 1/x) / (1 - 1/x)$

Solution:

Let $f(x) = \frac{1+\frac{1}{x}}{1-\frac{1}{x}} = \frac{\frac{x+1}{x}}{\frac{x-1}{x}} = \frac{x+1}{x-1}$, where $x \neq 0$

Using quotient rule, we get

$$f'(x) = \frac{(x-1)\frac{d}{dx}(x+1) - (x+1)\frac{d}{dx}(x-1)}{(x-1)^2}, x \neq 0,1$$

Further, we get

$$= \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2}, x \neq 0,1$$

So,

$$= \frac{x-1-x-1}{(x-1)^2} x \neq 0,1$$

$$= \frac{-2}{(x-1)^2} x \neq 0,1$$

Question 7

$1/(ax^2 + bx + c)$

Solution:

Let f (x) $=\frac{1}{ax^2+bx+c}$

Using quotient rule, we get

$$f'(x) = \frac{(ax^2+bx+c)\frac{d}{dx}(1) - \frac{d}{dx}(ax^2+bx+c)}{(ax^2+bx+c)^2}$$

By further calculation, we get

$$= \frac{(ax^2+bx+c)(0) - (2ax+b)}{(ax^2+bx+c)^2}$$

$$= \frac{-(2ax+b)}{(ax^2+bx+c)^2}$$

Question 8

(ax + b) / px² + qx + r

Solution:

Let f (x) $= \frac{ax+b}{px^2+qx+r}$

Using quotient rule, we get

$$f'(x) = \frac{(px^2+qx+r)\frac{d}{dx}(ax+b) - (ax+b)\frac{d}{dx}(px^2+qx+r)}{(px^2+qx+r)^2}$$

Further we get

$$= \frac{(px^2+qx+r)(a) - (ax+b)(2px+q)}{(px^2+qx+r)^2}$$

Again, by further calculation, we get

$$= \frac{apx^2+aqx+ar+ -2apx^2-aqx-2bpx-bq}{(px^2+qx+r)^2}$$

$$= \frac{-apx^2-2bpx-ar-bq}{(px^2+qx+r)^2}$$

Question 9

(px² + qx + r) / ax + b

Solution:

Let f (x) $= \frac{px^2+qx+r}{ax+b}$

Using quotient rule, we get

$$f'(x) = \frac{(ax+b)\frac{d}{dx}(px^2+qx+r) - (px^2+qx+r)\frac{d}{dx}(ax+b)}{(ax+b)^2}$$

By further calculation, we get

$$= \frac{(ax+b)(2px+q) - (px^2+qx+r)(a)}{(ax+b)^2}$$

So, we get

$$= \frac{2apx^2+aqx+2bpx+bq - apx^2 - aqx - ar}{(ax+b)^2}$$

$$= \frac{apx^2+2bpx+bq-ar}{(ax+b)^2}$$

Question 10

(a / x⁴) – (b / x²) + cox x

Solution:

Let $f(x) = \frac{a}{x^4} - \frac{b}{x^2} + \cos x$

By differentiating we get

$$f'(x) = \frac{d}{dx}\left(\frac{a}{x^4}\right) - \frac{d}{dx} + \left(\frac{d}{x^2}\right)\frac{d}{dx}(\cos x)$$

On further calculation, we get

$$= a\frac{d}{dx}(x^{-4}) - b\frac{d}{dx}(x^{-2}) + \frac{d}{dx}(\cos x)$$

We know that

$$\left[\frac{d}{dx}(x^n) = nx^{n-1} \text{ and } \frac{d}{dx}(\cos x) = -\sin x\right]$$

So,

$$= a(-4x^{-5}) - b(-2x^{-3}) + (-\sin x)$$

$$= \frac{-4a}{x^5} + \frac{2b}{x^3} - \sin x$$

Question 11

$4\sqrt{x} - 2$

Solution:

Let $f(x) = 4\sqrt{x} - 2$

By differentiating we get,

$$f'(x) = \frac{d}{dx}(4\sqrt{x} - 2) = \frac{d}{dx}(4\sqrt{x}) - \frac{d}{dx}(2)$$

Further, we get

$= 4\ \frac{d}{dx}\left(x^{\frac{1}{2}}\right) - 0$

$= 4\ \left(\frac{1}{2}x^{\frac{1}{2}-1}\right)$

$= \left(2x^{\frac{1}{2}}\right)$

$= \frac{2}{\sqrt{x}}$

Question 12

(ax + b) ^{n}

Solution:

Let f (x) = (ax + b) ^{n}

Accordingly, $f\ (x+h) = \{a\ (x+b) + b\}^n = (ax + ah + b)^n$

Using first principal, we get

$f^{'}(x) = \lim_{h\to 0}\frac{f\ (x+h) - f\ (x)}{h}$

$= \lim_{h\to 0}\frac{(ax+ah+b)^n\ (ax+b)^n}{h}$

Further we get,

$= \lim_{h\to 0}\frac{(ax+b)^n\left(1+\frac{ah}{ax+h}\right)^n - (ax+b)^n}{h}$

$= (ax+b)^n\lim_{h\to 0}\frac{\left(1+\frac{ah}{ax+h}\right)^n - 1}{h}$

By using binominal theorem, we get

$= (ax+b)^n\lim_{h\to 0}\frac{1}{h}\left[\left\{1+n\left(\frac{ah}{ax+b}\right)+\frac{n\ (n-1)}{2}\left(\frac{ah}{ax-b}\right)^2+\cdots.\right\}-1\right]$

Now, we get

$= (ax+b)^n\lim_{h\to 0}\frac{1}{h}\left[n\left(\frac{ah}{ax+b}\right)+\frac{n\ (n-1)a^2h^2}{\lfloor 2\ (ax+b)^2}+\cdots..(\text{Termscontaining higher degrees of h})\right]$

So, we get

$= (ax+b)^n\lim_{h\to 0}\left[\frac{na}{ax+b}+\frac{n\ (n-1)a^2h}{\lfloor 2\ (ax+b)^2}+\cdots\right]$

On further calculation we get

$= (ax+b)^n\left[\frac{na}{(ax+b)}+0\right]$

$= na \dfrac{(ax+b)^n}{(ax+b)}$
$= na\ (ax+b)^{n-1}$

Question 13

(ax + b) ⁿ (cx + d) ᵐ

Solution:

Let f (x) = $(ax+b)^n(cx+d)^m$
By using Leibnitz product rule, we get
$$f\ (x) = (ax+b)^n \frac{d}{dx}(cx+b)^n + (cx+b)^m \frac{d}{dx}(ax+b)^n$$
Let $f_1(x) = (ax+b)^m$
Then, $f_1(x+h) = (cx+ch+d)^m$
$$f'_1(x) = \lim_{h\to 0}\frac{f_1(x+h) - f_1(x)}{h}$$
$$\lim_{h\to 0}\frac{(cx+ch+d)^m - (cx+d)^m}{h}$$
By taking $(cx+d)^m$ as common, we get
$$= (cx+d)^m \lim_{h\to 0}\frac{1}{h}\left[\left(1+\frac{mch}{(cx+d)}+\frac{m\,(m-1)}{2}\frac{(c^2h^2)}{(cx+d)^2}+\cdots..\right)...1\right]$$
Now, we get
$$= (cx+d)^m \lim_{h\to 0}\frac{1}{h}\left[\frac{mch}{(cx+d)}+\frac{m\,(m-1)c^2h^2}{2\,(cx+d)^2}+\cdots..\ \text{(Terms containing higher degrees of h)}\right]$$
We know that,
$$\frac{d}{dx}(cx+d)^m = mc\,(cx+d)^{m-1}$$
Similarly, $\dfrac{d}{dx}(ax+b)^n = na\,(ax+b)^{n-1}$
$$= (cx+d)^m \lim_{h\to 0}\left[\frac{mch}{(cx+d)}+\frac{m\,(m-1)c^2h^2}{2\,(cx+d)^2}+\cdots..\right]$$
Now, we get
$$= (cx+d)^m\left[\frac{mc}{cx+d}+0\right]$$
$$= \frac{mc(cx+d)^m}{(cx+d)}$$
$$= mc\,(cx+d)^{m-1}$$
Hence, we get
$$f'(x) = (ax+b)^n\{mc(cx+d)^{m-1}\} + (cx+d)^m\{na\,(ax+b)^{n-1}\}$$
$$= (ax+b)^{n-1}(cx+d)^{m-1}[mc\,(ax+b) + na\,(cx+d)]$$

Question 14

Sin (x + a)

Solution:

Let f (x) = sin (x + a)
f (x + h) = sin (x + h + a)
By using first principal, we get
f' (x) = $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h \to 0} \frac{\sin(x+h+a) - \sin(x+a)}{h}$$

On further calculation, we get

$$= \lim_{h \to 0} \frac{1}{h}\left[2\cos\left(\frac{x+h+a+x+a}{2}\right)\sin\left(\frac{x+h+a-x-a}{2}\right)\right]$$

So, we get

$$= \lim_{h \to 0} \frac{1}{h}\left[2\cos\left(\frac{2x+2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\cos\left(\frac{2x+2a+h}{2}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}\right]$$

By taking limits we get

$$= \lim_{h \to 0} \cos\left(\frac{2x+2a+h}{2}\right) \lim_{\frac{h}{2} \to 0}\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}$$

Hence, we get

$$= \cos\left(\frac{2x+2a}{2}\right) \times 1$$

$$= \cos(x+a)$$

Question 15

cosec x cot x

Solution:

Let f (x) cosec x cot x
By using first principal, we get
f (x) = cosec x (cot x)' + cot x (cosec)' ……………. (1)
Let $f'_1(x) = \cot x$
Accordingly, $f'_1(x+h) = \cot(x+h)$
By using first principal we get

$$f'_1(x) = \lim_{h\to 0}\frac{f'_1(x+h) - f'_1\ (x)}{h}$$

$$= \lim_{h\to 0}\frac{\cot(x+h) - \cot x}{h}$$

On further calculation we get

$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x}\right)$$

Now we get

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x\cos(x+h) - \cos x\sin(x+h)}{\sin x\sin x\ (x+h)}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(\sin -x - x - h)}{\sin x\sin(x+h)}\right]$$

We get

$$= \frac{1}{\sin x}\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(-h)}{\sin(x+h)}\right]$$

$$= \frac{1}{\sin x}\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{1}{\sin(x+h)}\right)$$

So, we get

$$= \frac{1}{\sin x}\left(\frac{1}{\sin(x+0)}\right)$$

$$= \frac{1}{\sin^2 x}$$

$$= -\,cosec^2 x$$

Hence, we get

(Cot x)' = $-\,cosec^2 x$ .... (2)

Now let $f_2(x) = \mathrm{cosec}\ x$ Accordingly $f_2(x+h) = \mathrm{cosec}\ (x+h)$

By using first principal, we get

$$f_2(x) = \lim_{h\to 0}\frac{f_2(x+h) - f_2\ (x)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}[cosec\ (x+h) - cosec\ x]$$

By calculation further, we get

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\sin(x+h)} - \frac{1}{\sin x}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x - \sin(x+h)}{\sin x\sin(x+h)}\right]$$

So,

$$= \frac{1}{\sin x}\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\sin(x-h)}\right]$$

$$= \frac{1}{\sin x} \lim_{h \to 0} \frac{1}{h} \left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\sin(x+h)} \right]$$

$$= \frac{1}{\sin x} \lim_{h \to 0} \left[\frac{-\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)} \right]$$

We get

$$= \frac{-1}{\sin x} \lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \lim_{h \to 0} \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}$$

$$= \frac{-1}{\sin x} \cdot 1 \cdot \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+0)}$$

$$= \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x}$$

$= -\text{cosec } x. \cot x$

Hence,

(cosec)' = - cosec x. cot x .... (3)

From equation (1) (2) and (3) we get

$f'(x) = cosec\, x\,(-cosec^2 x) + \cot x\,(-cosec\, x \cot x)$

$= -\text{cosec}^2 x - \cot^2 x \text{ cosex } x$

Question 16

$$\mathbf{\frac{\cos x}{1 + \sin x}}$$

Solution:

Let f (x) $\frac{\cos x}{1 + \sin x}$

By using quotient rule we get

$$f'(x) = \frac{(1+\sin x)\frac{d}{dx}(\cos x) - (\cos x)\frac{d}{dx}(1+\sin x)}{(1+sin)^2}$$

$$= \frac{(1+\sin x)(-\sin x) - (\cos x)(\cos x)}{(1+sin)^2}$$

We get

$$= \frac{-\sin x - sin^2 x - cos^2 x}{(1+sin)^2}$$

$$= \frac{-\sin x - (sin^2 x + cos^2 x)}{(1+sin)^2}$$

Now, we get

$= \dfrac{-\sin x - 1}{(1+\sin)^2}$

$= \dfrac{-(1+\sin x)}{(1+sin)^2}$

$= \dfrac{-1}{(1+\sin x)}$

Question 17

$\dfrac{\boldsymbol{sin\ x + cos\ x}}{\boldsymbol{sin\ x - cos}}$

Solution:

Let f (x) = $\frac{sin\ x+cos\ x}{sin\ x-cos}$

By differentiating and using quotient rule, we get

$$f'(x) = \frac{(\sin x - \cos x)\ \frac{d}{dx}\ (\sin x + \cos x) - (\sin x + \cos x)\ \frac{d}{dx}\ (\sin x - \cos x)}{(\sin x - \cos x)^2}$$

On further calculation, we get

$$= \frac{(\sin x - \cos x)(\cos x - \sin x) - \ (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$$

$$= \frac{-(\sin x - \cos x)^2 - \ (\sin x + \cos x)^2}{(\sin x - \cos x)^2}$$

By expanding the terms we get

$$= \frac{[sin^2\ x + \ cos^2\ x - 2\sin x\cos x + \ sin^2\ x + \ cos^2\ x + 2\sin x\cos x]}{(\sin x - \cos x)^2}$$

We get

$$= \frac{1\ [1+1]}{(\sin x - \cos x)^2}$$

$$= \frac{-2}{(\sin x - \cos x)^2}$$

Question 18

$\dfrac{\sec x - 1}{\sec x + 1}$

Solution:

Let $f'(\text{x}) \dfrac{\sec x - 1}{\sec x + 1}$

Now this can be written as

$$f'(x)\frac{\frac{1}{\cos x}-1}{\frac{1}{\cos x}+1}=\frac{1-\cos x}{1+\cos x}$$

By differentiating and using quotient rule, we get'

$$f'(x)=\frac{(1+\cos x)\frac{d}{dx}(1-\cos x)-(1-\cos x)\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$=\frac{(1+cos\ x)(sin\ x)-\ (1-cos\ x)(-\ sin\ x)}{(1+\cos x)^2}$$

On multiplying we get

$$=\frac{\sin x+\cos x\sin x+\sin x-\sin x\cos x}{(1+\cos x)^2}$$

$$=\frac{2\sin x}{(1+\cos x)^2}$$

This can be written as

$$=\frac{2\sin x}{\left(1+\frac{1}{\sec x}\right)^2}$$

On taking L.C.M. we get

$$=\frac{2\sin x}{\frac{(\sec x+1)^2}{sec^2 x}}$$

On further calculation, we get

$$=\frac{2\sin x\ sin^2 x}{(\sec x+1)^2}$$

$$=\frac{\frac{2\sin x}{\cos x}\sec x}{(\sec x+1)^2}$$

$$=\frac{2\sec x\tan x}{(\sec x+1)^2}$$

Question 19

$\sin^n x$

Solution:

Let y = $\sin^n x$

Accordingly, for n = 1, y = sin x

We know that,

$$\frac{d}{dx}=\cos x, \text{i.e.} \frac{d}{dx}\sin x=\cos x$$

For n = 2, y = $\sin^2 x$

So, $\frac{d}{dx}=\frac{d}{dx}(\sin x\sin x)$

By Leibnitz product rule, we get

$= (\sin x)' \sin x + \sin x\, (\sin x)'$

$= \cos x \sin x + \sin x \cos x$

$= 2 \sin x \cos x$ .... (1)

For n = 3, y = $\sin^3$ x

So, $\frac{dy}{dx} = \frac{d}{dx}(\sin x \sin^2 x)$

By Leibnitz product rule, we get

$= (\sin x)' \sin^2 + \sin x\, (\sin^2 x)'$

From equation (1) we get

$= \cos x\, sin^2 x + \sin x\, (2 \sin x \cos x)$

$= \cos x\, sin^2 x + 2sin^{\,2} x \cos x$

$= \ 3\, sin^2 x \cos x$

We state that, $\frac{d}{dx}(sin^n\ x) = n\, sin^{(n-1)}\ x \cos x$

For n = k, let our assertion be true

i. e. $\frac{d}{dx}(sin^k\ x) = k\, sin^{(k-1)}\ x \cos x$ .... (2)

Now consider

$\frac{d}{dx}(sin^{k+1}\ x) = \frac{d}{dx}(\sin sin^k\ x)$

By using Leibnitz product rule, we get

$= (\sin x)'\ sin^k\ x + \sin x\, (sin^k\ x)'$

From equation (2) we get

$= \cos x\, sin^k\ x + \sin x\, \left(k\, sin^{(k-1)} x \cos x\right)$

$= \cos x\, sin^k\ x\, ksin^k\ x \cos x$

$= \ (k+1) sin^k\ x \cos x$

Hence, our assertion is true for n = k + 1

Therefore, by mathematical induction, $\frac{d}{dx}(sin^n\ x) = n\, sin^{(n-1)}\ x \cos x$

Question 20

$$\frac{a + b \sin x}{c + d \cos x}$$

Solution:

Let f (x) = $\frac{a+b \sin x}{c+d \cos x}$

By differentiating and using quotient rule, we get

$$f^{'}(x) = \frac{(c + d \cos x)\frac{d}{dx}(a + b \sin x) - (a + b \sin x)\frac{d}{dx}(c + d \cos x)}{(c + d \cos x)^2}$$

$$= \frac{(c + d \cos x)(b \cos x) - (a + b \sin x)(- d \sin x)}{(c + d \cos x)^2}$$

On multiplying we get

$$= \frac{cb\cos x + ad\,cos^2 x + ad\sin x + bd\,sin^2 x}{(c + d\cos x)^2}$$

Now taking bd as common we get

$$= \frac{bc\cos x + ad\sin x + bd\,(cos^2 x + sin^2 x)}{(c + d\cos x)^2}$$

$$= \frac{bc\cos x + ad\sin x + bd}{(c + d\cos x)^2}$$

Question 21

$$\frac{\sin(x + a)}{\cos x}$$

Solution:

Let $f(x) = \frac{sin\ (x+a)}{cos\ x}$

By differentiating and using quotient rule. We get

$$f'(x) = \frac{\cos x\,\frac{d}{dx}[\sin(x + a)] - \sin(x + a)\frac{d}{dx}\cos x}{cos^2 x}$$

$$f'(x) = \frac{\cos x\,\frac{d}{dx}[\sin(x+a)] - \sin(x+a)(-\sin x)}{cos^2 x} \quad \text{......... (i)}$$

Let g (x) = sin (x + a) Accordingly, g (x + h) = sin (x + h + a)

$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$

$= \lim_{h \to 0} \frac{1}{h}[\sin(x + h + a) - \sin(x + a)]$

On further calculation we get

$$= \lim_{h \to 0} \frac{1}{h}\left[2\cos\left(\frac{x + h + a + x + a}{2}\right)\sin\left(\frac{x + h + a - x - a}{2}\right)\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[2\cos\left(\frac{2x + 2a + h}{2}\right)\sin\left(\frac{1}{h}\right)\right]$$

$$= \lim_{h \to 0}\left[2\cos\left(\frac{2x + 2a + h}{2}\right)\left\{\frac{\sin\left(\frac{1}{h}\right)}{\left(\frac{1}{h}\right)}\right\}\right]$$

Now, taking limits we get

$$= \lim_{h \to 0}\cos\left(\frac{2x + 2a + h}{2}\right)\lim_{\frac{1}{h} \to 0}\left\{\frac{\sin\left(\frac{1}{h}\right)}{\left(\frac{1}{h}\right)}\right\}\left[\text{As h} \to \frac{h}{2} \to 0\right]$$

We know that

$$\left[\lim_{h \to 0}\frac{\sin h}{h} = 1\right]$$

$= \left(\cos\frac{2x+2a}{2}\right) \times 1$

$= \cos(x+a)$ ……………… (ii)

From equation (i) an (ii) we get

$f'(x) = \frac{\cos x.\cos(x+a) + \sin\ x\ sin(x+a)}{cos^2\ x}$

$= \frac{\cos(x+a-x)}{cos^2\ x}$

$= \frac{\cos a}{cos^2\ x}$

Question 22

x^4 (5 sin x – 3 cos x)

Solution:

Let f (x) = x^4 (5 sin x – 3 cos x)

By differentiating and using product rule, we get

$f'(x) = x^4\frac{d}{dx}(5\sin x - 3\cos x) + (5\sin x - 3\cos x)\frac{d}{dx}(x^4)$

On further calculation we get

$= x^4\left[5\frac{d}{dx}(\sin x) - 3\frac{d}{dx}(\cos x)\right] + (5\sin x - 3\cos x)\frac{d}{dx}(x^4)$

So, we get

$= x^4[5\cos x - 3(-\sin x)] + (5\sin x - 3\cos x)(4x^3)$

By taking x^3 as common we get

$= x^3[5x\cos x + 3x\sin x + 20\sin x - 12\cos x]$

Question 23

(x^2 + 1) cos x

Solution:

Let f (x) = (x^2 + 1) cos x

By differentiating and using product rule, we get

$f'(x) = (x^2+1)\frac{d}{dx}(cos\ x) + cos\ x\ \frac{d}{dx}(x^2+1)$

On further calculation, we get

$= (x^2+1)(-\sin x) + \cos x\ (2x)$

By multiplying we get

$= -x^2\sin x - \sin x + 2\ x\ cos\ x$

Question 24

$(ax^2 + \sin x)(p + q\cos x)$

Solution:

Let f (x) $(ax^2 + \sin x)(p + q\cos x)$
By differentiating and using product rule, we get
Let $f(x) = (ax^2 + \sin x)\frac{d}{dx}(p + q\cos x) + (p + q\cos x)\frac{d}{dx}(ax^2 + \sin x)$
On further calculation, we get
$= (ax^2 + \sin x) - (-q\sin x) + (p + q\cos x) + (2ax + \cos x)$
$= -q\sin x\ (ax^2 + \sin x) + (p + q\cos x) + (2ax + \cos x)$

Question 25

$(x + \cos x)(x - \tan x)$

Solution:

Let f (x) = (x + cos x) (x – tan x)
By differentiating and using product rule, we get

$$f'(x) = (x + \cos x)\frac{d}{dx}(x - \tan x) + (x - \tan x)\frac{d}{dx}(x + \cos x)$$

$$= (x + \cos x)\left[\frac{d}{dx}(x) - \frac{d}{dx}(\tan x)\right] + (x\tan x)(1 - \sin x)$$

Now, we get

$$= (x + \cos x)\left[1 - \frac{d}{dx}\tan x\right] + (x\tan x)(1 - \sin x)$$

Let g (x) = tan x . Accordingly g (x + h) = tan (x + h)
By using first principal we get

$g'(x)\ \lim_{h\to 0}\frac{g(x+h)- (x)}{h}$

$$\lim_{h\to 0}\left(\frac{\tan(x+h) - \tan x}{h}\right)$$

On further calculation, we get

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x - \sin x\cos(x+h)}{\cos(x+h)\cos x}\right]$$

Now we get

$$= \frac{1}{\cos x}\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)}\right]$$

$= \frac{1}{\cos x} \lim_{h \to 0} \frac{1}{h}\left[\frac{\sin h}{\cos(x+h)}\right]$

So, we get

$= \frac{1}{\cos x}\left(\lim_{h \to 0} \frac{\sin h}{h}\right)\left(\lim_{h \to 0} \frac{1}{\cos(x+h)}\right)$

We get

$= \frac{1}{\cos x}.1.\frac{1}{\cos(x+0)}$

$= \frac{1}{\cos^2 x}$

$= \sec^2 x$ ………… (ii)

Hence from equation (i) and (ii) we get

$f'(x) = (x+\cos x)(1 - \sec^2 x) + (x - \tan x)(1 - \sin x)$

$= (x+\cos x)(-\tan^2 x) + (x - \tan x)(1 - \sin x)$

$= -\tan^2 x\,(x+\cos x) + (x - \tan x)(1 - \sin x)$

Question 26

$$\frac{4x + 5\sin x}{3x + 7\cos x}$$

Solution:

Let f (x) $\frac{4x+5\sin x}{3x+7\cos x}$

By differentiating and using product rule, we get

$$f(x) = \frac{(3x+7\cos x)\frac{d}{dx}(4x+5\sin x) - (4x+5\sin x)\frac{d}{dx}(3x+7\cos x)}{(3x+7\cos x)^2}$$

On further calculation we get

$$= \frac{(3x+7\cos x)\left[4\frac{d}{dx}(x) + 5\frac{d}{dx}(\sin x)\right] - (4x+5\sin x)\left[3\frac{d}{dx}x + 7\frac{d}{dx}\cos x\right]}{(3x+7\cos x)^2}$$

$$= \frac{(3x+7\cos x)(4x+5\sin x) - (4x++5\sin x)(3-7\sin x)}{(3x+7\cos x)^2}$$

$$= \frac{12x + 15x\cos x + 28\cos x + 35\cos^2 x - 12x + 28x\sin x - 15\sin x\,35\sin^2 x}{(3x+7\cos x)^2}$$

We get

$$= \frac{15x\cos x + 28\cos x + 28x\sin x - 15\sin x + 35(\cos^2 x + \sin^2 x)}{(3x+7\cos x)^2}$$

$$= \frac{35 + 15x\cos z + 28\cos x + 28x\sin x - 15\sin x}{(3x+7\cos x)^2}$$

Question 27

$$\frac{\mathbf{x^2 \cos\left(\frac{\pi}{4}\right)}}{\mathbf{\sin x}}$$

Solution:

Let f (x) $\frac{x^2 cos\left(\frac{\pi}{4}\right)}{sinx}$

By differentiating and using product rule, we get

$$f\ (x) = \cos\frac{\pi}{4}.\left[\frac{\sin x\ \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(\sin x)}{\sin^2 x}\right]$$

By further calculation we get

$$= \cos\frac{\pi}{4}.\left[\frac{\sin x . 2\,x - x^2 \cos x}{\sin^2 x}\right]$$

By taking x as common, we get

$$= \frac{x \cos\frac{\pi}{4}[2 \sin x - x \cos x]}{sin^2\ x}$$

Question 28

$$\frac{x}{1 + tan\ x}$$

Solution:

Let f (x) $\frac{x}{1 + tan\ x}$

By differentiating and using product rule, we get

$$f'(x) = \frac{(1 + \tan x)\frac{d}{dx}(x) - x\ \frac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2}$$

$$f'(x) = \frac{(1+\tan\ x\ ) - x\ .\frac{d}{dx}(1+\tan\ x)}{(1+\tan\ x)^2} \qquad \text{............ (i)}$$

Let g (x) = 1 + tan x. Accordingly, g (x + h) = 1 tan x (x + h)

$$g'(x) = \lim_{h \to 0}\frac{g\ (x + h) - g\ (x)}{h}$$

$$= \lim_{h \to 0}\left[\frac{1 + \tan(x + h) - 1 - \tan x}{h}\right]$$

$$= \lim_{h \to 0}\frac{1}{h}\left[\frac{\sin\ (x + h)}{\cos\ (x + h)} - \frac{\sin x}{\cos x}\right]$$

By taking L.C.M. we get

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\sin(x+h)\cos x - \sin x\cos(x+h)}{\cos(x+h)\cos x}\right]$$

We get

$$\lim_{h \to 0} \frac{1}{h}\left[\frac{\sin(x+h+x)}{\cos(x+h)\cos x}\right]$$

$$\lim_{h \to 0} \frac{1}{h}\left[\frac{\sin h}{\cos(x+h)\cos x}\right]$$

So, we get

$$= \left(\lim_{h \to 0} \frac{\sin h}{h}\right) . \left(\lim_{h \to 0} \frac{1}{\cos(x+h)\cos x}\right)$$

$$= 1 \times \frac{1}{cos^2 x} = sec^2 x$$

$$\frac{d}{dx}(1 \tan x) = sec^2 x \qquad \text{.............. (ii)}$$

From equation (i) and (ii) we get

$$f'(x) = \frac{1 \tan x - x\, sec^2 x}{sec^2 x}$$

Question 29

(x + sec x) (x – tan x)

Solution:

Let $f(x)$(x + sec x) (x – tan x)

By differentiating and using product rule, we get

$$f'(x) = (\text{x} + \sec \text{x})\frac{d}{dx}(\text{x} - \tan \text{x}) + (\text{x} - \tan \text{x})\frac{d}{dx}(\text{x} - \sec \text{x})$$

So, we get

$$= (\text{x} - \sec \text{x})\left[\frac{d}{dx}(x) - \frac{d}{dx}\, tan\, x\right] + (x - tan\, x)\left[\frac{d}{dx}(x) + \frac{d}{dx}\, sec\, x\right]$$

$$= (\text{x} - \sec \text{x})\left[1 - \frac{d}{dx}\tan x\right] + (x - \tan x)\left[1 + \frac{d}{dx}\sec x\right] \qquad \text{........... (i)}$$

Let $f_1 = \tan x, f_2(x) = \sec x$

Accordingly, $f_1(x+h) = \tan(x+h) and\ f_2(x+h) = \sec(x+h)$

$$f_1(x) = \lim_{h \to 0}\left(\frac{f_1(x+h) - f_1(x)}{h}\right)$$

$$= \lim_{h \to 0}\left(\frac{tan\,(x+h) - tan}{h}\right)$$

By further calculation, we get

$$= \lim_{h \to 0}\left[\frac{\tan(x+h) - \tan x}{h}\right]$$

$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{sinx}{\cos x}\right]$

Now, by taking L.C.M. we get

$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\sin(x+h)\cos x - \sin x \cos(x-h)}{\cos(x+h)\cos x}\right]$

$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right]$

$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\sin h}{\cos(x+h)\cos x}\right]$

$= \left(\lim_{h \to 0} \frac{\sin h}{h}\right).\left(\lim_{h \to 0} \frac{1}{\cos(x+h)\cos x}\right)$

$= 1 \times \frac{1}{cos^2 x} = sec^2 x$

Hence, we get

$\frac{d}{dx}\tan x = sec^2\, x$………… (ii)

Now take

$f'_2(x) = \lim_{h \to 0}\left(\frac{f'_2(x+h) - f'_2(x)}{h}\right)$

$= \lim_{h \to 0}\left(\frac{\sec(x+h) - \sec x}{h}\right)$

This can be written as

$= \lim_{h \to 0}\left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x}\right]$

By taking L.C.M. we get

$\lim_{h \to 0} \frac{1}{h}\left[\frac{\cos x - \cos(x+h)}{\cos(x+h)\cos x}\right]$

On further calculation, we get

$= \frac{1}{\cos x} . \lim_{h \to 0} \frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right) . \sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right]$

$= \frac{1}{\cos x} . \lim_{h \to 0} \frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right) . \sin\left(\frac{-h}{2}\right)}{\cos(x+h)}\right]$

We get

$= \frac{1}{\cos x} . \lim_{h \to 0}\left[\frac{\sin\left(\frac{2x+h}{2}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right\}}{\cos(x+h)}\right]$

By taking limits, we get

$$= \sec x \,.\, \frac{\left\{\lim_{h\to 0} \sin\left(\frac{2x+h}{2}\right)\right\}\left\{\lim_{\frac{h}{2}\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right\}}{\lim_{h\to 0} \cos(x+h)}$$

We get

$$= \sec x \, \frac{\sin x \,.1}{\cos x}$$

$$\frac{d}{dx} \sec x = \sec x \tan x \quad \text{.................. (iii)}$$

From equation (i) (ii) and (iii) we get

$$f'(x) = (x + \sec x)\,(1 - sec^2 x) + (x + \tan x)\,(1 + \sec x \tan x)$$

Question 30

$$\frac{\mathbf{x}}{\mathbf{\sin^n x}}$$

Solution:

$$\text{Let } f(x) = \frac{x}{\sin^n x}$$

By differentiating and using product rule, we get

$$f'(x) = \frac{\sin^n x \frac{d}{dx} x - x \frac{d}{dx} \sin^n x}{\sin^{2n} x}$$

Easily, it can be shown that

$$\frac{d}{dx} \sin^n x = n\sin^{n-1} x \cos x$$

Hence,

$$f'(x) = \frac{\sin^n x \frac{d}{dx} x - x \frac{d}{dx} \sin^n x}{\sin^{2n} x}$$

By further calculation we get

$$= \frac{\sin^n x.\,1 - x\,(n \sin^{n-1} x \cos x)}{\sin^{2n} x}$$

By taking common terms we get

$$= \frac{\sin^{n-1} x.\,1 - x\,(\sin x - nx \cos x)}{\sin^{2n} x}$$

Hence, we get

$$= \frac{\sin x - nx \cos x}{\sin^{n+1} x}$$